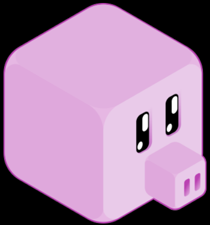
VISST Summer Camp

Black Holes for Hackers

David Wakeham

About me

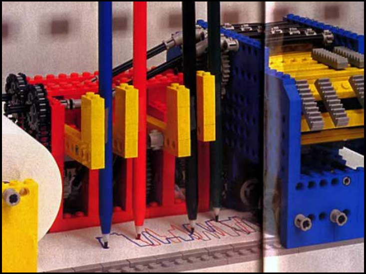


- Born and raised in Melbourne, Australia. Hence the accent!
- I moved to Vancouver to do PhD on black holes and string theory. I also ran the UBC Physics Circle for a few years!
- I now work on machine learning for quantum computers.

0. Overview

What is hacking?

- Hacking can refer to breaking security systems. But back in the day, it meant a playful, creative approach to technical matters.
- Examples: MIT pranks such as turning a building into Tetris,



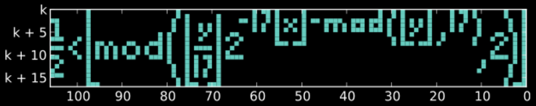
or the lego printer made by Google CEO Larry Page.

What is a hack?

- A hack is an act of playful technical ingenuity. Like a joke!

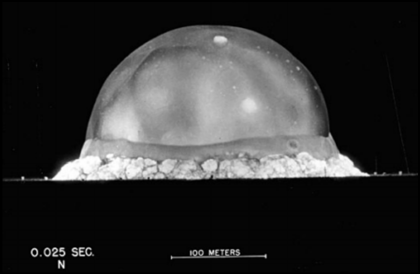
> *[Hackers] wanted to be able to do something in a more exciting way than anyone believed possible and show 'Look how wonderful this is. I bet you didn't believe this could be done.'*
>
> RICHARD STALLMAN

- Another example: Tupper's formula, which graphs itself!



Hacking physics

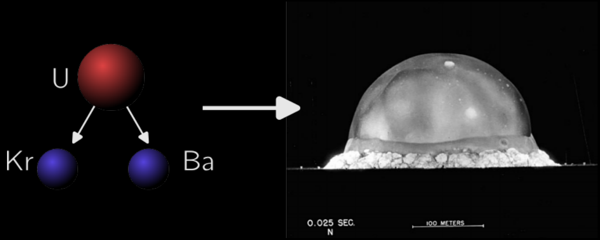
- We can hack physics with the same attitude!
- Example: the first atomic bomb test, aka the Trinity Test.



- Although the energy yield was classified, physicist G. I. Taylor calculated it from the picture. This is an amazing hack!

Day 1: Energy

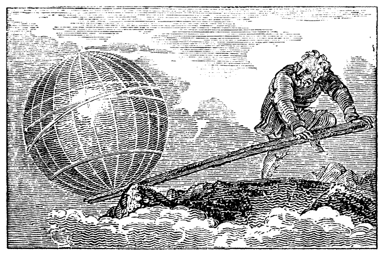
- This brings us to today's main theme: energy.
- Energy is a thing that is conserved, but comes in different forms. By changing forms, it causes change in physical systems.



- The atom bomb converts mass into energy yield via $E = mc^2$.

Day 2: Dimensional analysis

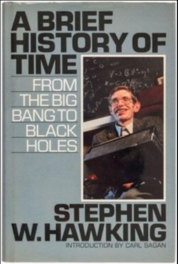
- Tomorrow, we'll introduce dimensional analysis, the technique Taylor used to estimate the energy.
- Although it uses only simple math, it is extraordinarily powerful.

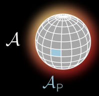


- It will be our main tool for hacking black holes!

Day 3: Black hole basics

- After covering these two topics (and reproducing Taylor's analysis of the atom bomb) we'll move onto black holes.
- We'll start with a brief history of these fascinating objects.

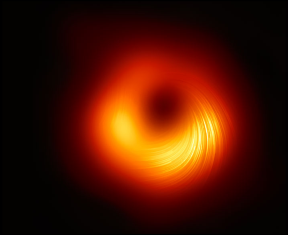


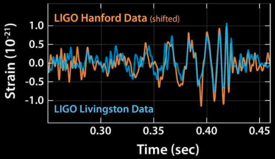


- We'll combine energy and dimensional analysis to learn various properties of black holes, including their temperature and area.

Day 4: Sound and vision

- The theoretical properties are all well and good, but how do we know black holes exist? They're hard to observe directly.
- Two methods: gravitational (LIGO) and radio waves (ETH).





- We'll discuss these methods, along with black hole collisions and how to turn the earth into a giant camera.

Day 5: Bonus round!

- On the last day we'll cover bonus material (time permitting).

- I've also prepared exercises to keep things interactive. We can work through these individually or together. It's up to you!

1. Energy

What is energy?

- Energy is a weird thing. It comes in a bunch of different flavors and it's hard to explain what ties them together.

- We'll be describing some of the flavours shortly.
- But what ties them together is that although the form can change, the total quantity doesn't change with time.

Feynman's blocks

- Richard Feynman described it in terms of building blocks:

  > *[A child] has blocks which are absolutely indestructible, and cannot be divided into pieces. Each is the same as the other. Let us suppose that he has 28 blocks. His mother puts him with his 28 blocks into a room at the beginning of the day. At the end of the day, being curious, she counts the blocks very carefully, and discovers a phenomenal law—no matter what he does with the blocks, there are always 28 remaining!*

- The blocks take on various forms, e.g. bumps under the rug, the displacement of bathwater, additional weight for a toy box.
- But once a conversion factor is found, she always counts 28.

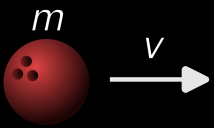
Conversion factors

- To put it differently, we start by counting blocks.
- Later, we find blocks can be converted into other quantities.

- A conversion factor tells us how to turn the other quantities into an equivalent number of blocks, so the total stays constant.

Vis viva

- The "building block" we use to count units of energy is called *vis viva* or kinetic energy, associated with motion.



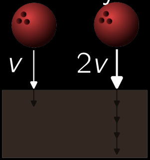
- If an object of mass m moves at speed v, the kinetic energy is

$$E_{\text{kin}} = \frac{1}{2}mv^2.$$

- We measure energy in joules (J), after James Joule.

Émilie's experiment

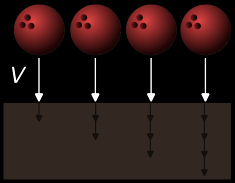
- There are a few obvious questions about this strange formula:
  - Why is it v^2 and not v?
  - Where does the factor of $1/2$ come from?
  - Is it obvious that this is conserved?
- The first question was answered by Émilie du Châtelet.



- She dropped brass spheres into clay, and observed that a sphere moving twice as fast burrows four times deeper.

Counting blocks

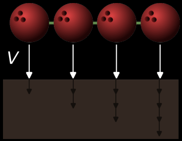
- The depth a ball burrows has a block-counting property: each unit of depth requires the same unit of effort.
- We could replace a ball moving at $2v$ with 4 balls moving at v.



- The effort yielded by the falling sphere is then proportional to v^2. (Of course, this all assumes effort doesn't depend on depth!)

Exercises: Energy 1

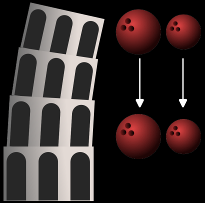
- **Exercise 1.1.** Is kinetic energy always conserved?
  - No! Consider two donuts which collide and stick together. They started with kinetic energy, and now they have none!
- **Exercise 1.2.** Give a concepual argument that the effort yielded should be proportional to m. *Hint*: split a ball into pieces.
  - Take the four balls in the picture and join them together.



We just argued that this yields four times the effort. But it's four times the mass! The argument clearly generalizes.

Galileo's experiment

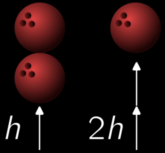
- A simpler experiment suggests the same conclusion.
- Galileo Galilei famously dropped iron balls of different weights from the Leaning Tower of Pisa. (So the legend goes.)



- They fell the same way, and more importantly for us, distance fallen was proportional to speed squared, $d \propto v^2$.

Weightlifting

- To drop the iron balls, he first had to lift them.
- Like burrowing into clay, lifting weights has a block-counting property, a fact later observed by Leibniz and Descartes.

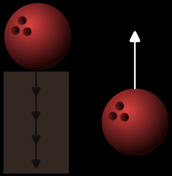


- The effort required to lift two equal masses a height h equals the effort to lift one mass $2h$. So effort is proportional to both:

$$\text{effort} \propto \text{mass} \times \text{height}.$$

Work

- The "effort" required to lift something, or burrow into clay, are evidently another form of energy. It is called work.
- In both cases, we push against resistance over some distance.

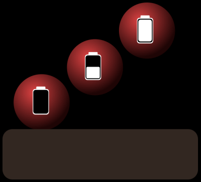


- The size of the push is called force. Work is the product of both:

$$\text{work} = \text{force} \times \text{distance}.$$

Potential energy

- For lifting, work has a special name: potential energy E_{pot}.
- The idea is that we can store energy by raising things. It's like a battery! By letting them fall, we recover the energy.



- The force is weight $F = mg$, where m is mass and $g \approx 9.8\ \text{m/s}^2$ is gravitational acceleration at the earth's surface, so

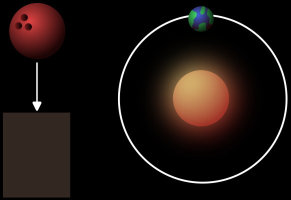
$$E_{\text{pot}} = mgh.$$

Exercises: Energy 2

- **Exercise 1.3.** We now have kinetic energy and work. Is their sum always conserved?
    - No! In the donut example, perhaps kinetic energy is converted into the work needed to press them together. But what happens to it afterwards? It's gone.
- **Exercise 1.4.** The idea of a battery is more than an analogy! Can you think of examples where gravity generates electricity?
    - Hydroelectricity! Falling water spins turbines. This is how we make energy right here in BC, hence BC Hydro.
- **Exercise 1.5.** Galileo found that speed and height fallen are related by $v^2 = 2gh$. Use this to explain the factor of $1/2$ in E_{kin}.
    - We have $E_{\text{pot}} = mgh = mv^2/2$. The factor comes from requiring potential and kinetic energy to be equal.

Heaven and earth

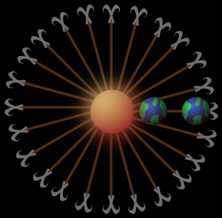
- Now things get interesting! Isaac Newton realized that a universal law of gravitation could explain heaven and earth.
- The same force makes things fall and the earth orbit the sun.



- This single law is universal in the sense that every object with mass is attracted to every other object with mass.

Inverse squares

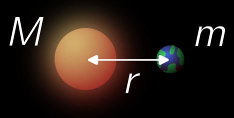
- This attraction falls as the inverse square of distance, $F \propto 1/r^2$.
- We can understand why by imagining each object shoots out grappling hooks in every direction. They attach to other masses.



- Since the grappling hooks fan out in a sphere with surface area $4\pi r^2$, a mass of fixed size can only absorb a fraction $\propto 1/r^2$.

Gravitational potential

- Suppose we have masses M and m. Since the attraction is universal, each part of M attracts each part of m.
- This means the attraction is proportional to both, $F \propto Mm$.

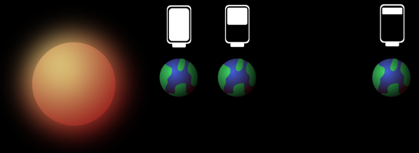


- We can store energy as before. Since work = force × distance, we might guess that the stored energy E_{grav} is

$$E_{\text{grav}} \propto \frac{Mm}{r^2} \times r = \frac{Mm}{r}.$$

Binding energy

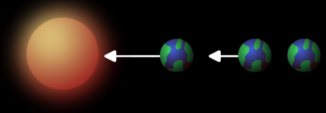
- This is almost right! But there is a problem: stored energy decreases with distance. This is opposite to what we expect!
- The solution is to put a minus sign in front, $E_{\text{grav}} \propto -Mm/r$.



- Negative energy may sound strange, but it just means an object is trapped. We need to pay a "binding energy" to free it!

Newton's constant

- The last ingredient is the conversion factor, turning gravitational energy into an equivalent amount of kinetic energy.
- Ideally, we'd drop a planet into the sun and track its motion.



- If we did, we'd find a factor called Newton's constant, G:

$$\frac{GMm}{r} = \frac{1}{2}mv^2.$$

But we can't do this! But there is an indirect way!

Exercises: Energy 3 (Newton's constant)

- **Exercise 1.6.** Consider an object of mass m sitting a height h above the earth, which has radius $R_\oplus$ and mass $M_\oplus$.
  (a) Show that the gravitational potential E_{grav} is

$$E_{\text{grav}} = -\frac{GMm}{R_\oplus + h}.$$

  (b) It is a mathematical fact that, for small x,

$$\frac{1}{1+x} \approx 1 - x.$$

  From this, argue that the potential is approximately

$$E_{\text{grav}} \approx -\frac{GMm}{R_\oplus} + \left(\frac{GM}{R_\oplus^2}\right) mh.$$

Exercises: Energy 3 (Newton's constant)

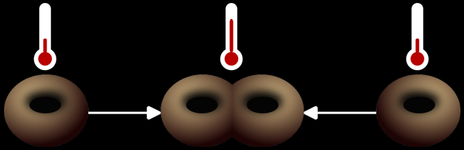
- We can connect this to the usual potential energy $E_{\text{pot}} = mgh$.
  (c) Both E_{pot} and E_{grav} measure gravitational potential. Even worse, they are obviously different! Show that energy conservation means they can only differ by a constant.
  (d) Removing this constant and equating what remains, argue that

$$g = \frac{GM_\oplus}{R_\oplus^2}.$$

  (e) We know from dropping things that $g \approx 9.8 \text{ m/s}^2$, and from shadows at the solstice that $R_\oplus \approx 6300$ km. The earth is mostly hot metal, with density 5500 kg/m^3. Estimate G. (You should get $G \approx 6.7 \times 10^{-11} \text{ m}^3/\text{kg s}^2$.)

Hot donuts

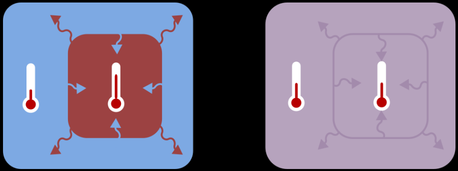
- Despite having various forms of energy—kinetic, gravitational, work—it's still not clear that energy is conserved.
- Two donuts smush into one. Where did the kinetic energy go?



- The answer is revealed if we measure their temperature. The missing energy becomes heat! The same is true for friction.

Giving energy away

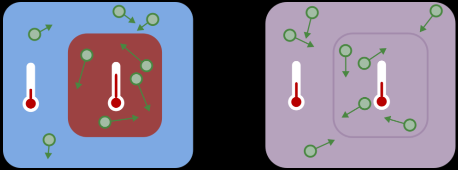
- What is temperature? There are two ways to think of it. The first is as as tendency to give energy to your surroundings.
- A hotter object will give more energy than it receives and cool.



- The cooler object receives more than it gives and heats up. The process stops at equilibrium, i.e. the same temperature!

Molecular motion

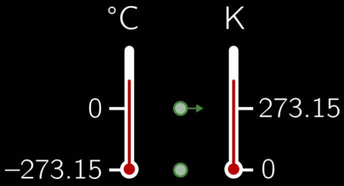
- This is the viewpoint of 19th century thermodynamics.
- In the 20th century, we realized heat is molecular motion! Temperature is then kinetic energy per molecule.



- The higher the temperature, the faster the molecules move! Their speed can vary, so temperature governs the average.

Stay positive

- To turn temperature into energy, we need to measure correctly!
- Kinetic energy is always positive. We need a temperature scale which stays positive! Neither Celsius or Fahrenheit will do.



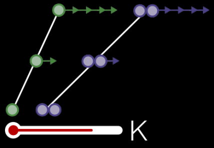
- We use Kelvin (K), where 0 K means molecules are still. Kelvin is related to Celsius by x K $= (x - 273.15)^\circ$ C.

Counting calories

- With Kelvin, we can convert temperature into kinetic energy.
- For T in Kelvin, the average KE per molecule is

$$\epsilon_{\text{kin}} = f k_B T,$$

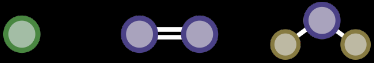
where k_B is a conversion factor called Boltzmann's constant and f counts the number of ways to move, rotate, or vibrate.



- For N total molecules, the thermal energy is $E_{\text{therm}} = fNk_B T$.

Exercises: Energy 4 (Boltzmann's constant)

- **Exercise 1.7.** In three-dimensional space, calculate f for (a) a xenon atom; (b) oxygen, O_2; and (c) water, H_2O. Note that rotations don't count unless they change how a molecule looks.



(a) The xenon atom can only move, so $f = 3$.

(b) The molecule can move three ways, rotate two ways perpendicular to the axis (rotating around the axis doesn't change how it looks), and vibrate along the axis, so $f = 3 + 2 + 1 = 7$.

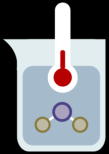
(c) This is a bit harder! But we can view each hydrogen bond separately. The bond can rotate in two dimensions, and oscillate in length. So each contributes $f = 3$. Since there are two bonds, and the full molecule can move, we have $f = 9$.

Exercises: Energy 4 (Boltzmann's constant)

- **Exercise 1.8.** It takes roughly 4.18 J of energy to raise the temperature of 1 mL of water by 1 K. Each molecule weighs

$$2.99 \times 10^{-23} \text{ g}.$$

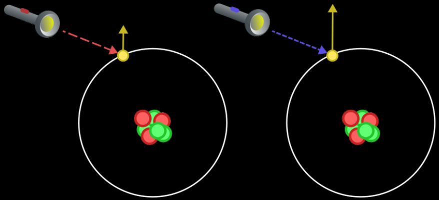
Using this information, and $f = 9$ from the last exercise, estimate Boltzmann's constant k_B.



- *Hint*: Since $E_{\text{therm}} = fNk_B T$, a change of 1 K requires fNk_B. (You should get $k_B \approx 1.39 \times 10^{-23}$ J/K.)

The photoelectric effect

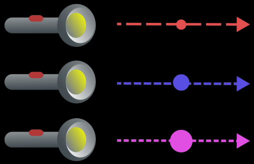
- We've almost finished our grand tour of forms of energy. We'll end with two forms proposed by ALBERT EINSTEIN in 1905.
- Electrons orbit the nucleus. Shine light, and they are released.



- This is called the photoelectric effect. It has two features:
  - the speed of ejected electrons depends only on frequency;
  - the number of ejected electrons depends only on brightness.

Photons

- Einstein proposed that light is made of individual particles called photons, which strike electrons and knock them out of the atom.



- Individual photons have energy depending on frequency $\nu = 1/\text{period}$ as

$$E_{\text{phot}} = h\nu,$$

where h is a conversion factor called Planck's constant.

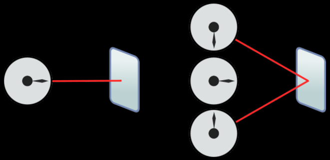
- Incidentally, this is the physics behind solar panels!

Exercises: Energy 5 (Planck's constant)

- **Exercise 1.9.** Let's try to determine Planck's constant! It's hard to observe individual electrons. So we use the photoelectric effect as a battery, and determine its voltage V. Then $mv^2/2 = eV$, where $e = 1.6 \times 10^{-19}$ C is the electron charge.
  (a) In the same way that masses can be bound by gravitational attraction, electrons are bound by attraction to the nucleus. If the binding energy is Φ, show ejected electrons have energy
  $$\frac{1}{2}mv^2 = eV = h\nu - \Phi.$$
  (b) In zinc, the minimum frequency required to liberate electrons is $\nu = 1.4 \times 10^{15}$ Hz. At $\nu = 2 \times 10^{15}$ Hz, it takes a voltage of $V = 2.5$ to stop the electrons. Use this to estimate h.
  (You should get $h \approx 6.63 \times 10^{-34}$ J/Hz.)

Relativity

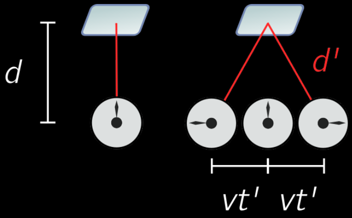
- We'll end with the most famous equation in physics..
- Einstein's theory of special relativity is based on the idea that, however you choose to measure, the speed of light is the same.



- We can measure the speed of light by bouncing a laser off a mirror. With moving clock, we get a different speed unless the clock slows down to compensate.

Time dilation

- Suppose the mirror is a vertical distance d from a stationary clock. The clock fires a laser, which returns in time $2t = 2d/c$.

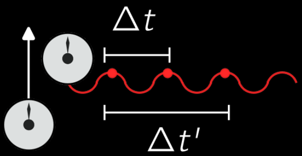


- A second clock travels horizontally at speed v. If lights reaches the mirror in time t', it travels $d' = \sqrt{d^2 + (vt')^2}$ per leg.
- Requiring $c = d'/t' = d/t$ implies the moving clock is slower by

$$t' = t\sqrt{1-(v/c)^2} = t/\gamma.$$

Frequency shifts

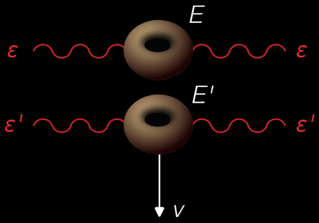
- We can view frequency as number of peaks per clock tick. This means a moving clock detects a higher frequency.
- Since $\Delta t = \Delta t'/\gamma$, the frequency increases as $\nu' = \gamma\nu$.



- From Einstein's formula $\varepsilon = h\nu$, this means $\varepsilon' = \gamma\varepsilon$. Like our mirror experiment, photons here are perpendicular to motion.

The laser donut argument

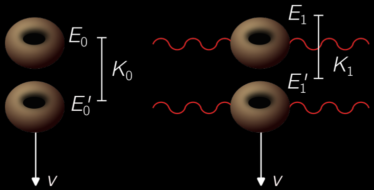
- Suppose an object, e.g. a donut, emits two photons of energy ε opposite ways. The donut has energy E_0 before and E_1 after.



- Conservation of energy means $E_0 = E_1 + 2\varepsilon$.
- Now repeat the experiment with a donut moving at speed v. We now have $E_0' = E_1' + 2\varepsilon'$, where $\varepsilon' = \gamma\varepsilon$ has increased.

Mass-energy

- Since the only difference is motion, $K_0 = E_0' - E_0$ and $K_1 = E_1' - E_1$ equal kinetic energy of the donut before and after.



- But before and after is not the same! In fact,

$$K_0 - K_1 = 2(\varepsilon' - \varepsilon) = 2\varepsilon(\gamma - 1).$$

The velocity of the donut is the same, so the mass has changed.

Exercises: Energy 6 (mass-energy)

- **Exercise 1.10.** Let's complete the argument that $E = mc^2$.
  (a) Show that, if an object travelling at fixed speed v changes kinetic energy by ΔK, the change in mass is $\Delta m = 2\Delta K / v^2$.
  (b) There is a mathematical result that, for small x,

$$\frac{1}{\sqrt{1-x}} \approx 1 + \frac{1}{2}x.$$

  From this, argue that $\Delta K = K_0 - K_1$ is approximately

$$\Delta K \approx \frac{\epsilon v^2}{c^2}.$$

  (c) Combine (a) and (b) to show $2\epsilon = (\Delta m)c^2$. So energy 2ϵ has mass $2\epsilon / c^2$! This led Einstein to his famous formula.

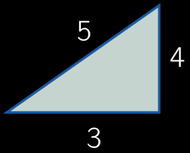
Recap

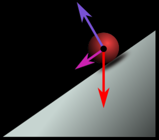
- Let's briefly recap. We've seen many forms of energy:
  1. kinetic energy, $E_{\text{kin}} = \frac{1}{2}mv^2$;
  2. work, $E_{\text{work}} = Fd$;
  3. potential energy, $E_{\text{pot}} = mgh$;
  4. gravitational binding energy, $E_{\text{grav}} = -\frac{GMm}{r}$;
  5. thermal energy, $E_{\text{therm}} = Nfk_B T$;
  6. light, $E_{\text{phot}} = h\nu$; and finally
  7. mass, $E = mc^2$.
- All of them are relevant to black holes! But to apply them, we will need to learn dimensional analysis.

2. Dimensional analysis

What is a dimension?

- Before diving into details, I'll start with the big picture.
- Math is about numbers. Physics is about measurements. These are what we obtain from performing experiments!





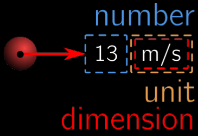
- A measurement is not only a number, but the physical aspect of the system our measurement probes. That's all a dimension is!

Anatomy of a measurement

- Measurements are packaged as numbers plus units, e.g.

$$v = 13 \text{ m/s}, \quad t = 48 \text{ hours}.$$

The dimension is, in some sense, within the unit.

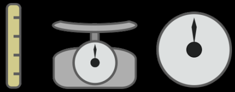


- To calculate dimension: (1) throw away the number and (2) ask the unit: what do you measure? E.g.

$$[v] = [\cancel{13} \text{ m/s}] = [\text{m/s}] = \text{speed}$$
$$[t] = [\cancel{48} \text{ hours}] = [\text{hours}] = \text{time}.$$

Basic dimensions

- The power of dimensional analysis involves boiling things down to a few basic dimensions which can express everything else.
- We start with length (L), mass (M) and time (T):



$$L \quad M \quad T$$

- These are measured by rulers, scales and clocks respectively. We can write other dimensions in terms of these, e.g.

$$\left[\frac{\mathrm{m}}{\mathrm{s}}\right] = \text{speed} = \frac{L}{T}.$$

Algebra of dimensions

- Dimensions obey simple algebraic rules.
- Example 1 (powers):

$$[1\ \mathrm{cm}^2] = [\mathrm{cm}^2] = [\mathrm{cm}]^2 = L^2.$$

- Example 2 (different dimensions):

$$\left[4\,\frac{\mathrm{m}^3}{\mathrm{s}}\right] = \left[\frac{\mathrm{m}^3}{\mathrm{s}}\right] = \frac{[\mathrm{m}]^3}{[\mathrm{s}]} = \frac{L^3}{T}.$$

- Example 3 (formulas):

$$[E] = \left[\frac{1}{2}mv^2\right] = [m][v]^2 = \frac{ML^2}{T^2}.$$

Exercises: Dimensions 1

- **Exercise 2.1.** Find the dimensions of force in terms of L, M, T. *Hint*: Remember work (= force × distance) is a form of energy.
  - From the formula for work, we know force = energy/distance. The dimension of energy is ML^2/T^2 and the dimension of distance is L. So the dimension of force is

$$[\text{force}] = \frac{[\text{energy}]}{[\text{distance}]} = \frac{ML^2}{T^2} \times \frac{1}{L} = \frac{ML}{T^2}.$$

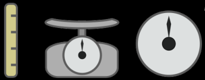
- **Exercise 2.2.** A megaparsec is a distance $\text{Mpc} = 3 \times 10^{19}$ km. The rate of cosmic expansion is usually quoted in terms of Hubble's constant, $H_0 = 70(\text{ km/s})/\text{Mpc}$. Find its dimension.
  - Since km and Mpc both measure distance, the dimensions cancel. Thus, $[H_0] = [1/s] = 1/T$.

Dimensional guesswork

- We can use physical laws to find dimensions, e.g. for energy.

$$W = Fd$$

dimensional analysis ↑   ↓ determining dimensions



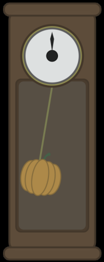
- Dimensional analysis reverses this process:

$$\text{dimensions} \implies \text{physical laws.}$$

We'll illustrate the general method for a pumpkin clock.

Pumpkin clock 1: setup

- Attach a pumpkin to a string and give it a small kick. It will starts to oscillate with some time period.

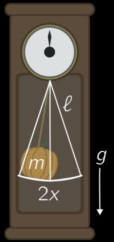


- Our goal is to find the time period using dimensional analysis, and keep time with the pumpkin.

Pumpkin clock 2: listing parameters

- We start by listing all the factors that could be relevant:

  1. the pumpkin mass m;
  2. the string length ℓ;
  3. the size of the swing, x;
  4. gravitational acceleration, g.



- Not all the parameters are relevant! Simple experiments show period does not depend on the amplitude of the swing.
- The moral: determining relevant quantities takes physics!

Pumpkin clock 3: putting it all together

- Now list dimensions for the remaining parameters:
  1. pumpkin mass $[m] = M$;
  2. string length $[\ell] = L$;
  3. finally, acceleration $[g] = [9.8 \text{ m/s}^2] = L/T^2$.
- The core assumption of dimensional analysis: the target quantity can be written as powers of relevant parameters. Here, we guess

$$t \sim m^a \ell^b g^c.$$

- To determine powers, take dimensions of both sides:

$$[t] = T, \quad [m^a \ell^b g^c] = \frac{M^a L^{b+c}}{T^{2c}}.$$

Pumpkin clock 4: solving for powers

- Now things get algebraic! To find a, b and c, we match dimensions on the LHS and RHS of

$$T = \frac{M^a L^{b+c}}{T^{2c}}.$$

	LHS	RHS
M	0	a
L	0	$b+c$
T	1	$-2c$

- This gives three equations for the three unknowns:

$$a = 0, \quad b + c = 0, \quad -2c = 1.$$

Pumpkin clock 5: pendulum period

- This is easily solved: $a = 0, b = -c = 1/2$. We now plug these numbers into our guess:

$$t \sim m^a \ell^b g^c = m^0 \ell^{1/2} g^{-1/2} = \sqrt{\frac{\ell}{g}}.$$

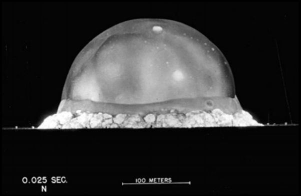
- This is almost right! The correct answer is $t = 2\pi\sqrt{\ell/g}$.
- This illustrates strengths and weaknesses of dimensional analysis:
  - $(-)$ We had to do an experiment to discard x.
  - $(-)$ We missed the factor of 2π.
  - $(+)$ We learned that m was irrelevant for free!
  - $(+)$ We're typically only off by "small" numbers!

Exercises: Dimensions 2

- **Exercise 2.3.** Explain why grandfather clocks are so large.
  - We want the period to be 2 seconds, i.e. a half-period is one tick. Then $2 = 2\pi\sqrt{\ell/9.8}$, or rearranging, $\ell = 9.8/\pi^2 \approx 1$ m. So they have to big enough to fit a meter-long pendulum!
- **Exercise 2.4.** Can you think of a physical reason that the mass *m* of the pumpkin might be irrelevant?
  - Recall that all objects fall the same way, regardless of mass. Since gravity causes the swinging, mass is irrelevant.
- **Exercise 2.5.** What happens with our pumpkin clock analysis if we use the angular velocity $\omega = 2\pi/t$ instead of the period t?
  - Everything is the same except that $[\omega] = 1/T$. Our analysis will result in $\omega = \sqrt{g/\ell}$. This is correct, including the factors!

The Trinity Test 1: parameters

- We are ready to hack the Trinity Test with dimensional analysis.



- To find the energy released E, we first list relevant factors:
  - time after detonation, t;
  - radius of detonation, R;
  - mass density of air, ρ; and
  - gravitational acceleration g.
- In fact, gravity isn't relevant in an explosion like this!

The Trinity Test 1: parameters

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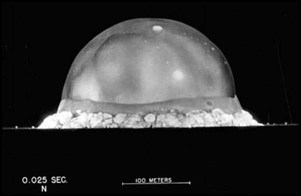

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  - radius of detonation, R;
  - mass density of air, ρ; and
  - ~~gravitational acceleration g.~~
- In fact, gravity isn't relevant in an explosion like this!

The Trinity Test 2: putting it all together

- Now we follow the same steps as the pumpkin clock. We find dimensions of relevant quantities:
    - time after detonation $[t] = T$;
    - radius of detonation $[R] = L$;
    - mass density of air $[\rho] = [1.29 \text{ kg}/\text{m}^3] = M/L^3$.
- Next, write the target as powers of relevant quantities:

$$E \sim t^a r^b \rho^c$$

and evaluate dimensions of LHS and RHS:

$$[E] = ML^2 T^{-2}, \quad [t^a r^b \rho^c] = T^a L^{b-3c} M^c.$$

The Trinity Test 3: solving for powers

- Equating LHS and RHS, we get equations for a, b, c:

$$T^{-2}L^2M = T^aL^{b-3c}M^c.$$

	LHS	RHS
M	1	c
L	2	$b-3c$
T	-2	a

- This gives three equations for the three unknowns:

$$a = -2, \quad b - 3c = 2, \quad c = 1.$$

I'll hand it over to you at this point!

Exercises: Dimensions 3 (Trinity Test)

- **Exercise 2.6.** Why do you think g is not a relevant quantity?
  - Gravity acts very slowly compared to a nuclear explosion! The mass of air, ρ, is relevant, since the resistance to movement is what determines the shape of the explosion.

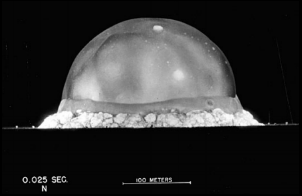
- **Exercise 2.7.** Solve the system of equations

$$a = -2, \quad b - 3c = 2, \quad c = 1$$

to obtain a dimensional guess $E \sim t^a R^b \rho^c$.

  - If we substitute $c = 1$ into $b - 3c = 2$, we get $b = 5$. So $a = -2, b = 5, c = 1$ and hence $E \sim t^{-2} R^5 \rho$.

Exercises: Dimensions 3 (Trinity Test)

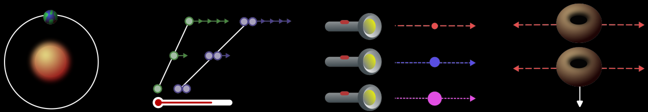


- **Exercise 2.8.** Use the estimate $E \sim t^{-2}R^5\rho$ along with the photo to estimate the energy yield in terms of kilotons of TNT (4.2×10^{12} J). You can take air density as $\rho \approx 1$ kg/m^3.
  - The photo tells us all we need to know! The time is indicated, $t = 0.025$ s, and $R \approx 140$ m. This yields

$$\frac{140^5 \times 1}{(0.025)^2(4.2 \times 10^{12})} \approx 20 \text{ kilotons.}$$

Revisiting energy

- Our first step will be to revisit forms of energy.



- Remember that we studied different, interconvertible types of energy. Four will be particularly important:
  - gravitational binding energy, $E_{\text{grav}} = -GMm/r$;
  - thermal energy, $E_{\text{therm}} = Nfk_{\text{B}}T$;
  - photon energy, $E_{\text{phot}} = h\nu$;
  - mass-energy, $E = mc^2$.

Fundamental constants

- Each form involves a constant that converts the terms into something with the same dimensions as kinetic energy.

$$G \quad k \quad h \quad c$$

- These are called the fundamental constants, since they tell us how different forms of energy are related.
- Photon energy is more often stated in terms of $\omega = 2\pi\nu$:

$$E_{\text{phot}} = h\nu = \frac{h}{2\pi} \cdot 2\pi\nu = \hbar\omega,$$

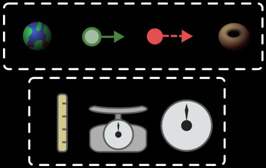
where $\hbar = h/2\pi$ is Planck's reduced constant. We'll use that!

Finding dimensions

- Remember that kinetic energy has dimensions

$$\left[\frac{1}{2}mv^2\right] = [m][v]^2 = \frac{ML^2}{T^2}.$$

- We can use this to find dimensions of fundamental constants.



- These are relevant quantities for dimensional analysis when that type of energy is involved! So the dimensions will be useful.

Einstein's energies

- We'll start with Einstein's energy formulas. The easiest is c, the speed of light. Since it is a speed, we have

$$[c] = \frac{L}{T}.$$

- Planck's constant is trickier. Since $\omega = 2\pi/\text{period}$,

$$\frac{ML^2}{T^2} = [E] = [\hbar\omega] = [\hbar] \times \frac{1}{T} \quad \Longrightarrow \quad [\hbar] = \frac{ML^2}{T}.$$

- I'll leave G for you to do!

Exercises: Dimensions 4

- **Exercise 2.9.** Compute the dimension of G.
  - From the dimension of energy and $E_{\text{grav}} = GMm/r$,

$$\left[\frac{GMm}{r}\right] = [G] \times \frac{M^2}{L} = \frac{ML^2}{T^2} \quad \Longrightarrow \quad [G] = \frac{L^3}{MT^2}.$$

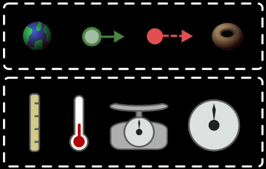
- **Exercise 2.10.** Show that $G\hbar/c^3$ has the dimensions of area, L^2.
  - Using our dimensions:

$$[G\hbar/c^3] = [G][\hbar][c]^{-3} = \frac{L^3}{MT} \times \frac{ML}{T} \times \frac{T^3}{L^3} = L^2.$$

  But area from the formula for the area of a rectangle, for instance, $[\text{area}] = [\text{width}][\text{height}] = L^2$.

New dimensions

- The remaining fundamental constant is Boltzmann's k_B. This will actually require a new fundamental dimension!



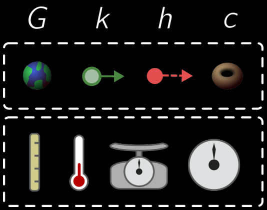
- This dimension is temperature Θ. Then

$$\frac{ML^2}{T^2} = [Nfk_B T] = [k_B T] = [k_B]\Theta \quad \Longrightarrow \quad [k_B] = \frac{ML^2}{\Theta T^2}.$$

Here, we discard f and N since they are pure numbers.

Physics with constants

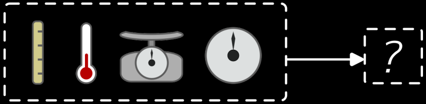
- We can now do physics with fundamental constants!



- Our dimensional analysis should include:
  1. Newton's constant $[G] = L^3/MT^2$, when things are heavy;
  2. Boltzmann's constant $[k_\mathrm{B}] = ML^2/\Theta T^2$, when things are hot;
  3. Planck's constant $[\hbar] = ML^2/T$, when things are quantum;
  4. and the speed of light $[c] = L/T$, when things are fast.

Recap

- Before moving on, let's recap how dimensional analysis works:
  1. list quantities that could be relevant;
  2. eliminate as many as possible from physical reasoning;
  3. find dimensions of the ones that remain;
  4. equate the dimension of powers to the target quantity;
  5. solve for the powers and you have your guess!



- Next, we will finally introduce black holes, and apply all the hard work we've done on energy and dimensional analysis.

Exercises: Dimensions 5 (Stefan-Boltzmann)

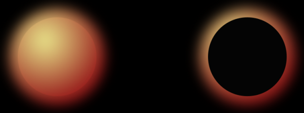
- **Exercise 2.11.** MAX PLANCK showed that hot objects glow due to the quantum excitation of particles on the surface. When they de-excite, they emit light. To analyze this process, we need a new dimension, $\mathcal{N}$, for the amount of matter on the surface.
  (a) Which fundamental constants should be involved?
  (b) Find the dimensions of the surface area $\mathcal{A}$ and total rate of energy emitted P using the new dimension $\mathcal{N}$.
  (d) Explain why the fundamental constants do not depend on $\mathcal{N}$.
  (c) Apply dimensional analysis to guess P, using $\mathcal{A}$ and the constants from (a). You should find that

$$P \sim \left(\frac{k_B^4}{\hbar^3 c^2}\right) \mathcal{A} T^4.$$

3. Black holes

Dark stars

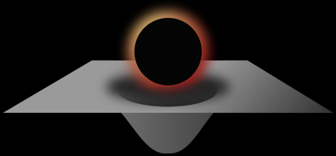
- We can finally start our voyage into black holes. Black holes actually began life under another name: dark stars.



- John Michell proposed "dark stars" way back in 1783.
- He realized that, according to Newtonian gravity, if light had mass, then very large stars could trap it! They'd become "dark".

General relativity

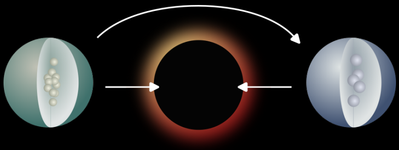
- In 1915, Einstein reinvented gravity. His theory of general relativity is about spacetime curvature rather than forces.



- A few months later, KARL SCHWARZSCHILD discovered that Einstein's theory allowed dark stars, at least in principle.
- But in practice, no one believed these objects could form.

Mounting evidence

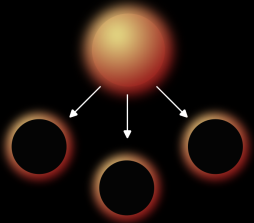
- But evidence began to mount they might really occur.
- In 1931, Subrahmanyan Chandrasekhar showed that, above a certain size, stars made of electrons would collapse.



- In principle, they could turn into neutrons. In 1939, Robert Oppenheimer showed that big neutron stars must collapse!

Inevitability

- Later that year, Einstein published a paper trying to prove black holes were impossible in general relativity. He was wrong!
- First, Oppenheimer found an explicit model of stellar collapse.



- After a 25 year lull, Roger Penrose and Stephen Hawking showed in the 60s that black holes are inevitable.

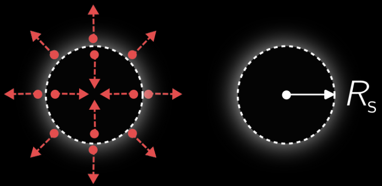
Black hole thermodynamics

- This paved the way for spectacular discoveries in the 70s.
- JAKOB BEKENSTEIN, Hawking, and others proposed that black holes have entropy and obey the laws of thermodynamcs.

- It was unclear if this was just an analogy. In 1974, Hawking showed black holes glow like a lump of coal. They're not black!

The Schwarzschild radius

- This concludes our history lesson. Let's do some physics!
- First, let's compute how big a black hole is, i.e. the size of the region where gravity traps light. The edge is the event horizon.



- For a sphere of matter, the event horizon is also a sphere. Its radius is called the Schwarzschild radius R_s.

The hackery begins

- Let's apply our recipe for dimensional analysis with constants.
- First, we list relevant quantities:
  - the mass m of the black hole;
  - Newton's constant G, since gravity is involved; and
  - the speed of light c, since light is involved.

  We'll assume the history of the black hole doesn't matter.
- The dimensions are:
  - $[m] = M$, of course;
  - $[G] = L^3/MT^2$, and
  - $[c] = L/T$, from our list.
- Finally, we write out target as powers, $R_s \sim m^a G^b c^d$.

Exercises: Black holes 1 (Schwarzschild radius)

- **Exercise 3.1.** Why is it reasonable to treat the black hole's history as irrelevant?
  - Features of the black hole, like its composition, shape, and so on, are hidden behind the event horizon. They can't be observed from outside, so shouldn't affect R_s.
- **Exercise 3.2.** Finish the analysis and show $R_s \sim Gm/c^2$.
  - Writing down dimensions, we have

$$L = [R_s] = [m]^a[G]^b[c]^d = M^a \times \frac{L^{3b}}{M^b T^{2b}} \times \frac{L^d}{T^d}.$$

We set $a = b$ to cancel mass, and $d = -2b$ to cancel time. This leaves $L^{3b+d} = L^b$, so we set $b = a = 1$ and $d = -2$.

Exercises: Black holes 1 (Schwarzschild radius)

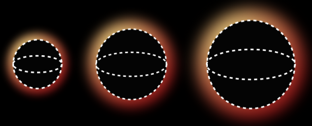
- **Exercise 3.3.** Michell actually calculated the Schwarzschild radius in 1783! (Perhaps it should be called the Michell radius.)
  (a) Suppose a particle of mass μ is a distance r from a black hole of mass m. How much energy does it need to escape?
  (b) If this energy is all given in the form of kinetic energy, show the particle must travel at
  $$v_{\text{esc}} = \sqrt{\frac{2Gm}{r}}.$$
  (c) Light travels at speed c. If light is a massive particle, argue it can no longer escape at radius $R_s = 2Gm/c^2$.
  (d) Why is it reasonable to view light as having a mass?

The area law

- The event horizon is a sphere with surface area

$$\mathcal{A} = 4\pi R_s^2 = \frac{16\pi G^2 m^2}{c^4}.$$

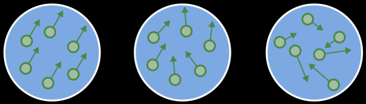
If it always swallows matter, m increases, and so does $\mathcal{A}$.



- This is for a spherical black hole. Hawking proved this "area law" in general, including for multiple colliding black holes.

Area and entropy

- Not many things just increase with time! An exception is entropy S, which goes up by the Second Law of Thermodynamics.



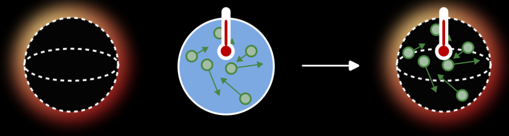
- Loosely speaking, entropy is related to thermal energy by

$$S = k_{\mathrm{B}} N f = \frac{E_{\mathrm{therm}}}{T}.$$

This led Bekenstein and Hawking to propose that $S_{\mathrm{BH}} \propto \mathcal{A}$.

Hawking radiation

- But is a black hole a real thermal system, or is this just an analogy? Bekenstein though the former, Hawking the latter.
- In 1974, Hawking got the biggest shock of his career.



- He found that, like a hot lump of coal, black holes have a temperature due to quantum effects at the surface. They glow!

A longlist

- Again, we can find the temperature with dimensional hackery. We start as usual by listing everything that could be relevant:
  - the mass $[m] = M$ of the black hole;
  - the Schwarzschild radius $[R_s] = L$;
  - Planck's constant $[\hbar] = ML^2/T$, since we have quantum effects;
  - Boltzmann's constant $[k_B] = ML^2/\Theta T^2$, since heat is involved;
  - the speed of light $[c] = L/T$, since radiation is light; and
  - Newton's constant $[G] = L^3/MT^2$, since gravity is involved.
- Unfortunately, this list is way too long! We need to prune.
- Since $R_s \sim Gm/c^2$, we don't need all of G, m and c as well. We fix size, so m and G (which determined R_s) aren't needed.

A shortlist

- Again, we can find the temperature with dimensional hackery. We start as usual by listing everything that could be relevant:
  - ~~the mass $[m] = M$ of the black hole;~~
  - the Schwarzschild radius $[R_s] = L$;
  - Planck's constant $[\hbar] = ML^2/T$, since we have quantum effects;
  - Boltzmann's constant $[k_B] = ML^2/\Theta T^2$, since heat is involved;
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- Unfortunately, this list is way too long! We need to prune.
- Since $R_s \sim Gm/c^2$, we don't need all of G, m and c as well. We fix size, so m and G (which determined R_s) aren't needed.

Hawking temperature: 1

- Our guess for the temperature is $T_{\mathrm{H}} \sim R_s^a k_B^b \hbar^d c^e$. The LHS has dimension Θ. The RHS has

$$L^a \cdot \left(\frac{ML^2}{\Theta T^2}\right)^b \cdot \left(\frac{ML^2}{T}\right)^d \cdot \left(\frac{L}{T}\right)^e = \frac{L^{a+2b+2d+e} M^{b+d}}{T^{2b+d+e}\Theta^b}.$$

This leads to four equations:

	LHS	RHS
L	0	$a + 2b + 2d + e$
M	0	$b + d$
T	0	$2b + d + e$
Θ	1	$-b$

Hawking temperature: 2

- We would like to solve the set of equations

	LHS	RHS
L	0	$a + 2b + 2d + e$
M	0	$b + d$
T	0	$2b + d + e$
Θ	1	$-b$

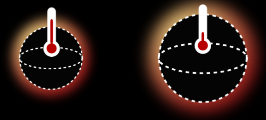
- Doing the math gives $a = b = -1, d = e = 1$. In detail:
  - The Θ equation tells us $b = -1$;
  - The M equation then tells us $d = 1$;
  - Subtracting T from L equation, $a + d = 0$;
  - Finally, since $d = 1$, $a = -1$.

Hawking temperature: 3

- Subbing $a = b = -1, d = e = 1$ into $T_{\mathrm{H}} \sim R_{\mathrm{s}}^a k^b \hbar^d c^e$ gives

$$T_{\mathrm{H}} \sim \frac{\hbar c}{k_{\mathrm{B}} R_{\mathrm{s}}} \sim \frac{\hbar c^3}{k_{\mathrm{B}} G m}.$$

- Up to a factor of $1/8\pi$, this agrees with Hawking's famous calculation of black hole temperature. It's a cool result!



- It is cool in another sense: large black holes are colder!

Exercises: Black holes 2 (evaporation)

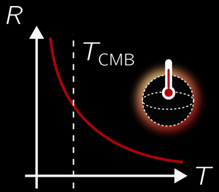
- **Exercise 3.4.** Black holes lose energy and hence evaporate.
  (a) Recall the Stefan-Boltzmann formula for rate of energy loss, $P \sim (k_B^4/\hbar^3 c^2)\mathcal{A}T_H^4$. Use this, along with the area $\mathcal{A} \sim R_s^2$, temperature $T_H \sim \hbar c/k_B Gm$, and total energy $E = mc^2$, to show that a black hole of mass m has approximate lifetime

$$t_{\text{evap}} \sim \frac{G^2 m^3}{\hbar c^4}.$$

  (b) The supermassive black hole at the centre of the galaxy, Sagittarius A*, has a mass of about 8.26×10^{36} kg. Roughly how long will it take to evaporate?

Exercises: Black holes 2 (evaporation)

- **Exercise 3.5.** The universe actually has some faint background warmth, left over from the big bang, called the cosmic microwave background. It is around $T_{\mathrm{CMB}} = 2.7$ K.

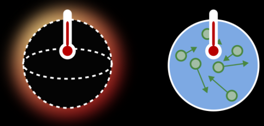


(a) If a black hole is in equilibrium with the CMB, then $T_{\mathrm{H}} = T_{\mathrm{CMB}}$. How big is such a black hole?

(b) Explain why this is the smallest a black hole can get in our current universe. *Hint:* What happens if it gets smaller?

Entropy revisited

- Remember that Hawking was motivated to think about black holes as thermal systems because area is like entropy.



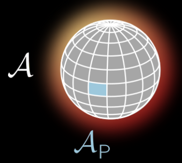
- We can check this! Assume thermal energy $E_{\text{therm}} \sim mc^2$. Then

$$S_{\text{BH}} = \frac{E_{\text{therm}}}{T_{\text{H}}} \sim \frac{mc^2}{\hbar c^3/k_{\text{B}}Gm} \sim \left(\frac{Gm}{c^2}\right)^2 \frac{k_{\text{B}}c^3}{G\hbar} \sim \frac{k_{\text{B}}\mathcal{A}}{G\hbar/c^3},$$

since $\mathcal{A} \sim R_s^2 \sim (Gm/c^2)^2$. So entropy is proportional to area.

Planck pixels

- We encountered $G\hbar/c^3$ earlier, when we showed that $[G\hbar/c^3] = L^2$. We call $\mathcal{A}_\mathrm{P} = G\hbar/c^3$ the Planck area.



- The horizon is a screen, and entropy counts the "Planck pixels":

$$S_\mathrm{BH} \sim \frac{k_\mathrm{B}\mathcal{A}}{\mathcal{A}_\mathrm{P}}.$$

- It's as if each pixel contributes a degree of freedom to the total Nf. This is a deep clue about the nature of spacetime!

Exercises: Black holes 3 (the Holographic Bound)

- **Exercise 3.6.** The Second Law states that the entropy of a closed system always increases. The Generalized Second Law (GSL) states that this remains true when we include black holes.
  (a) Let $m_{\mathcal{A}}$ be the mass of the spherical black hole of surface area $\mathcal{A}$. Explain why, if any spherical system of surface area $\mathcal{A}$ reaches mass $m_{\mathcal{A}}$, it should collapse to form a black hole.
  (b) Suppose we have an ordinary spherical system with surface area $\mathcal{A}$. Explain how we can add a shell of matter to force it to collapse into a black hole of area $\mathcal{A}$.
  (c) Imagine we have an ordinary sphere of matter with surface area $\mathcal{A}$ and entropy $S > S_{\text{BH}}$, greater than the associated black hole. Using (b), argue that this violates the GSL.

  This deep result is called the Holographic Bound.