String perturbation theory and Riemann surfaces

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Abstract

We derive an expression for the genus g partition function of the closed bosonic string, also called the *string measure*. Physically, these represent loop corrections to the vacuum amplitude. The derivation uses tensor calculus on Riemann surfaces, to which we give a self-contained introduction.

Contents

1. Introduction

1.1. **Invitation.** In quantum field theory (QFT), the basic objects are point particles. As they propagate through space and interact with each other, they describe a graph called a Feynman diagram. If the interactions are weak, we can calculate scattering amplitudes *perturbatively*, considering only Feynman diagrams up to some order in the interaction strength. Not all Feynman diagrams are different: they can be related by symmetries of the graphs themselves (*graph automorphisms*), and we shouldn't overcount.

In string theory, the basic objects are 1-dimensional rather than 0-dimensional. They can propagate through space and interact, just like point particles, but instead of sweeping out a graph they sweep out a *surface*. Just like perturbation theory in QFT, string perturbation theory involves summing over distinct surfaces. There are various ways for surfaces to be distinct. They could be *topologically distinct* (which will correspond to different orders in the interaction parameter), or homoeomorphic and differ in their rigid structure, that is, their geometry.

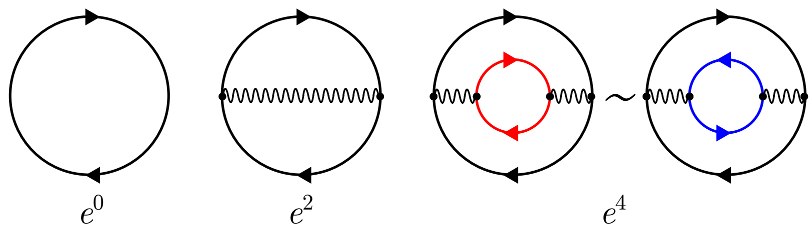

Figure 1. Some vacuum bubbles in QED at different orders in e. The last two diagrams are naively different, but related by a graph symmetry.

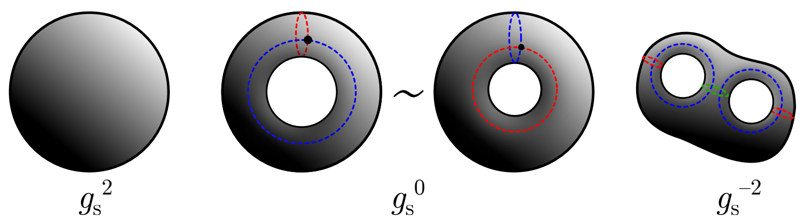

Figure 2. Some vacuum diagrams in closed bosonic string theory at different orders in g_s. The second and third diagrams are naively different, but related by a diffeomorphism (Dehn twist).

As with Feynman diagrams, apparently different geometries can correspond to the same diagram in perturbation theory. They can be connected by symmetries of the surface itself (*diffeomorphisms*) or gauge symmetries of the theory, since the metric on the surface is now a physical field. For a given homeomorphism class, the set of physically distinct surfaces is called the *moduli space*. Understanding the moduli space, and calculating its contribution to scattering amplitudes, involves considerably more mathematical machinery than the point-particle case. These notes aim to provide a brief and non-rigorous introduction to some of these tools.

1.2. **Background.** We begin with some background string theory material, focusing on closed bosonic strings. The *worldsheet* Σ is a two-dimensional manifold swept out by a closed loop (or loops) as time evolves. We will treat it as Euclidean from the outset. It has local coordinates $(\sigma^\alpha) = (\sigma, \tau)$, an auxiliary metric $\gamma_{\alpha\beta}$, and supports some scalar fields $\{X\} : \Sigma \to M$ which map the worldsheet to a target manifold M. For simplicity, we assume $M = \mathbb{R}^{1,D-1}$ is flat, D-dimensional Minkoswki space, with metric $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, \ldots, +1)$ and fields X^μ.

The dynamics of the theory is governed by the *Polyakov action*. Loosely speaking, this is just gravity on the worldsheet minimally coupled to the scalars X^μ:

$$S_{\mathrm{P}}[X, \gamma] := \frac{T}{2} \int d^2\sigma \, \sqrt{\gamma}\gamma^{\alpha\beta}\eta_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu + \frac{\lambda}{4\pi} \int d^2\sigma \, \sqrt{\gamma}\mathcal{R}^{(2)}$$

$$:= S_{\mathrm{P}}^X[X, \gamma] + \lambda\chi(\Sigma), \tag{1}$$

where $\gamma := |\det \gamma|$, $\mathcal{R}^{(2)}$ is the worldsheet Ricci scalar, and $T := 1/2\pi\alpha'$ is the *string tension*. We will discuss the coupling constant λ below. In two dimensions, the Einstein-Hilbert term is not genuinely dynamical. Using the *Gauss-Bonnet theorem*, we have rewritten it on the second line of (1) in terms of a topological invariant of the worldsheet, namely the *Euler characteristic*

$$\chi(\Sigma) := 2 - 2g - b, \tag{2}$$

where b is the number of boundaries and g the number of holes in the worldsheet. See Fig. 3 for an example. In the case of closed strings, b will simply be the number of asymptotic string states in a scattering process.

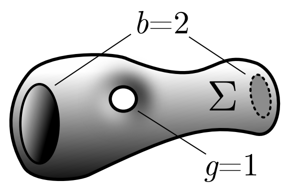

Figure 3. A worldsheet with two boundaries and one hole. It has Euler characteristic $\chi = 2 - 2g - b = -2$.

The Polyakov action (1) possesses two important *gauge redundancies*. These are not physical symmetries, but rather, different descriptions of the same physics. First, *Weyl invariance* under a rescaling of the metric:

$$\begin{aligned} \gamma_{\alpha\beta} &\to \tilde{\gamma}_{\alpha\beta} = e^{2\omega(\sigma)}\gamma_{\alpha\beta} \\ \delta\gamma_{\alpha\beta} &= 2\omega(\sigma)\gamma_{\alpha\beta}. \end{aligned} \tag{3}$$

Second, *diffeomorphism invariance* under a change of worldsheet coordinates:

$$\begin{aligned} \sigma^\alpha &\to \tilde{\sigma}^\alpha(\sigma), \quad \gamma_{\alpha\beta}(\sigma) \to \tilde{\gamma}_{\alpha\beta}(\tilde{\sigma}) = \frac{\partial\sigma^\lambda}{\partial\tilde{\sigma}^\alpha}\frac{\partial\sigma^\delta}{\partial\tilde{\sigma}^\beta}\gamma_{\lambda\delta}(\sigma) \\ \delta\sigma^\alpha &= -\epsilon^\alpha, \qquad\quad \delta\gamma_{\alpha\beta} = \nabla_\alpha\epsilon_\beta + \nabla_\beta\epsilon_\alpha. \end{aligned} \tag{4}$$

In both cases, we have written the infinitesimal version as well. For a worldsheet Σ, we denote the corresponding symmetry groups by $\mathrm{Weyl}(\Sigma)$ and $\mathrm{Diff}(\Sigma)$ respectively, and the gauge group is a *semidirect* product

$$G := \mathrm{Weyl}(\Sigma) \rtimes \mathrm{Diff}(\Sigma). \tag{5}$$

Note that if we fix the form of the metric, the gauge symmetry is now the *conformal group*: diffeomorphisms that can be undone by a Weyl transformation.[1]

To quantise the theory, the most elegant method is the *path integral*. Morally speaking, this is just a sum over classical field configurations weighted by e^{-iS}. For a Euclidean worldsheet, the scattering amplitude for b asymptotic string states $\{\alpha\}$, created by vertex operators $\{V_\alpha\}$, is given by the path integral

$$\begin{aligned} \mathcal{A}^{(b)}(\{\alpha\}) &= \int \mathcal{D}X\,\mathcal{D}\gamma\, e^{-S_{\mathrm{P}}[X,\gamma]}\prod_\alpha V_\alpha \\ &= \sum_{g\geq 0} e^{-\lambda(2-2g-b)}\frac{1}{\mathrm{vol}(G_{g,b})}\int_{\mathrm{Map}(\Sigma_{g,b},M)\times\mathrm{Met}(\Sigma_{g,b})\times\mathcal{V}_{g,b}} \mathcal{D}X\,\mathcal{D}\gamma\, e^{-S_{\mathrm{P}}^X[X,\gamma]}\prod_\alpha V_\alpha, \end{aligned} \tag{6}$$

[1] Incidentally, this overlap is the reason we need to use a *semidirect* product.

where $\mathrm{Map}(\Sigma_{g,b}, M)$ denotes the space of embeddings, $\mathrm{Met}(\Sigma_{g,b})$ the space of metrics on a worldsheet $\Sigma_{g,b}$ with g handles and b boundaries, and $\mathcal{V}_{g,b}$ is the space of vertex insertions (which we will henceforth ignore).

Schematically, $\mathrm{vol}(G_{g,b})$ is the volume of orbits of the gauge group G on $\mathrm{Map}(\Sigma_{g,b}, M) \times \mathrm{Met}(\Sigma_{g,b})$; we will discuss this more in §3.1. This is the same as directly integrating over the gauge orbits themselves,

$$\frac{\mathrm{Map}(\Sigma_{g,b}, M) \times \mathrm{Met}(\Sigma_{g,b})}{G} := \mathcal{E}_{g,b} \times \mathcal{M}_{g,b}, \tag{7}$$

where $\mathcal{M}_{g,b} := \mathrm{Mod}(\Sigma_{g,b})/G$ is the *moduli space* of physically distinct metrics described earlier, and $\mathcal{E}_{g,b} := \mathrm{Map}(\Sigma_{g,b}, M)/G$.

We will focus specifically on the *vacuum amplitude* $Z := \mathcal{A}^{(0)}$ where there are no asymptotic string states. Setting $b = 0$ in (6), we can write

$$Z := \sum_{g\geq 0} Z_g = \sum_{g\geq 0} g_{\mathrm{s}}^{2-2g} \int_{\mathcal{E}_g \times \mathcal{M}_g} \mathcal{D}X\, \mathcal{D}\gamma\, e^{-S_{\mathrm{P}}^X[X,\gamma]}. \tag{8}$$

The full vacuum amplitude Z is a sum over "loop amplitudes" Z_g at different orders in the *string coupling constant* $g_{\mathrm{s}} := e^{-\lambda}$. If g_{s} is small, this is a genuine perturbation expansion for the vacuum amplitude; the first few terms are depicted in Fig. 2. Our goal will be to write an explicit expression for Z_g in terms of certain differential operators on the worldsheet.

2. Complex geometry

2.1. **Teichmüller and MCG.** Let's look at the symmetries of Σ in a little more detail. First, consider the subgroup of "small" diffeomorphisms of Σ, that is, the identity component $\mathrm{Diff}_0(\Sigma)$. Restricting the gauge group $G_0 := \mathrm{Diff}_0(\Sigma) \rtimes \mathrm{Weyl}(\Sigma)$ to these small diffeomorphisms, and taking the quotient, yields the *Teichmüller space* of Σ:

$$\mathrm{Teich}(\Sigma_g) := \frac{\mathrm{Met}(\Sigma_g)}{G_0}. \tag{9}$$

Teichmüller space is finite-dimensional, with the dimension depending on genus:

$$\dim_{\mathbb{R}} \mathrm{Teich}(\Sigma_g) = \begin{cases} 0 & g = 0 \\ 2 & g = 1 \\ -3\chi(g) & g \geq 2. \end{cases} \tag{10}$$

This follows from the *Riemann-Roch theorem* (22), as we will show below. For the torus ($g = 1$), the Tëichmuller space is just the set of allowed complex parameters τ, namely the upper half-plane $\mathbb{H} := \{\tau : \Im(\tau) > 0\}$. This is indeed two-dimensional.

The components of $\mathrm{Diff}(\Sigma)$ form the *mapping class group*,

$$\mathrm{MCG}(\Sigma) := \frac{\mathrm{Diff}(\Sigma)}{\mathrm{Diff}_0(\Sigma)} = \frac{G}{G_0} = \frac{\mathrm{Teich}(\Sigma)}{\mathcal{M}_g}. \tag{11}$$

Viewed a different way, the Teichmüller space Teich is the *universal covering space* for $\mathcal{M}_g = \mathrm{Teich}/\mathrm{MCG}$: they share the same local structure (including real dimension), but while $\mathrm{Teich}(\Sigma_g)$ is simply connected, $\mathcal{M}_g$ is topologically nontrivial due to the MCG quotient. We can think of the MCG as the remaining "topological" gauge redundancy after locally fixing conformal gauge. See Fig. 4 for a cartoon of the different groups and quotients involved.

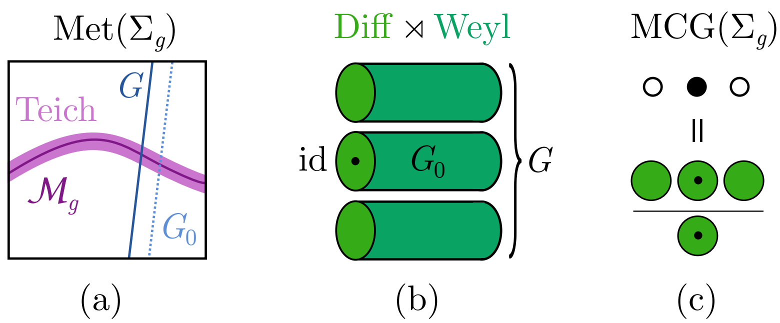

Figure 4. (a) The space of metrics is fibred by G, with base space $\mathcal{M}_g$. Equivalently, it is fibred by a coarser group G_0 with larger base space Teich. (b) A cartoon of the gauge group G, with diffeomorphisms split into components, and the identity marked. (c) The MCG is the quotient of Diff by Diff_0.

In the case of the torus, for instance, the mapping class group is $\text{PSL}(2, \mathbb{Z})$, generated by the *Dehn twists* S and T around the periods of the torus. In general, the MCG is generated (non-minimally) by the $3g - 1$ Dehn twists around red, blue and green loops pictured in Fig. 5(a). To perform a Dehn twist around a loop $\gamma \subset \Sigma$, we excise a small neighbourhood of γ, twist one of the boundaries by 2π, then reglue. The procedure is illustrated in Fig. 5(b). We can explicitly calculate the action of Dehn twists on the *first homology class* $H_1(\Sigma; \mathbb{Z})$, generated by the red and blue loops [1], but this will be unnecessary for our purposes.

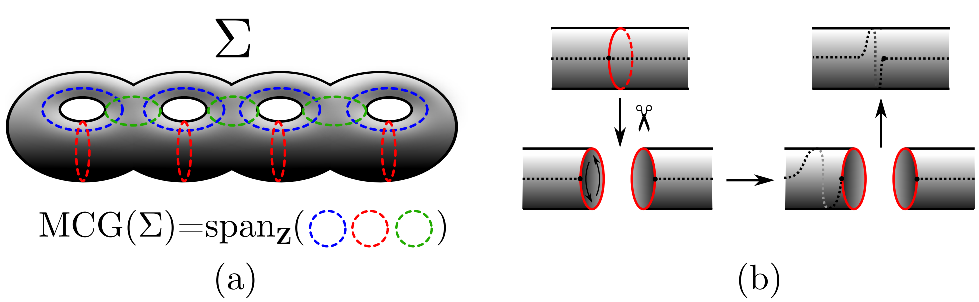

Figure 5. (a) A Riemann surface of genus $g = 4$. The red and blue loops provide a basis for the homology group H_1, while the Dehn twists around the red, blue and green loops generate the mapping class group $\text{MCG}(\Sigma)$. (b) A Dehn twist around the red loop. First cut, then twist by 2π, then glue.

2.2. **Calculus on Riemann surfaces.** Let Σ_g be a Euclidean worldsheet of genus g. On any local coordinate chart, we can choose the conformal gauge

$$\text{d}s^2 = e^{2\omega(\sigma)}(d\sigma^2 + d\tau^2). \qquad (12)$$

Recall that we can introduce complex coordinates, $z := \sigma + i\tau$, $\bar{z} := \sigma - i\tau$, with corresponding 1-forms

$$\mathrm{d}z = \mathrm{d}\sigma + i\mathrm{d}\tau, \quad \mathrm{d}\bar{z} = \mathrm{d}\sigma - i\mathrm{d}\tau, \tag{13}$$

vectors (or derivative operators)

$$\partial = \frac{1}{2}(\partial_\sigma - i\partial_\tau), \quad \bar{\partial} = \frac{1}{2}(\partial_\sigma + i\partial_\tau). \tag{14}$$

In complex coordinates, the conformal metric takes the form

$$\mathrm{d}s^2 = \frac{1}{2}e^{2\omega(z,\bar{z})} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{15}$$

On an overlapping chart with coordinates $w, \bar{w}$, we have an off-diagonal metric of the form (15) just in case the transition function is holomorphic, with invertible $w = w(z), \bar{w} = \bar{w}(\bar{z})$. This means that Σ_g is not only a real 2-manifold, but a *Riemann surface*.

As usual, we use the metric (15) to raise and lower indices. Since it is off-diagonal, we can always raise or lower a $\bar{z}$-index to obtain a z-index, and we can restrict ourselves to tensor fields with z indices only. Under a coordinate transformation $z \to w(z)$, vectors V and one-forms θ transform as

$$V^z \to V^w = \partial_z w V^z, \quad \theta_z \to \theta_w = (\partial_z w)^{-1}\theta_z.$$

Thus, for a tensor $T^{z\cdots z}_{z\cdots z}$ with n upstairs indices and m downstairs indices, built from the tensor product of vectors and one-forms, we have

$$T^{z\cdots z}_{z\cdots z} \to T^{w\cdots w}_{w\cdots w} = (\partial_z w)^{n-m}\, T^{z\cdots z}_{z\cdots z}. \tag{16}$$

The difference $n - m$ is called the *helicity* of the field. We denote the set of helicity k tensor fields on Σ by $\mathcal{H}^{(k)}(\Sigma)$.

We can differentiate these fields using the covariant derivatives $\nabla_z, \nabla_{\bar{z}}$. However, if we want to ensure our differentiated fields only have z indices, we should consider $\nabla^z = \gamma^{z\bar{z}}\nabla_{\bar{z}}$. Note that, from the index structure, ∇_z maps $\mathcal{H}^{(k)}(\Sigma) \to \mathcal{H}^{(k-1)}(\Sigma)$, while ∇^z maps $\mathcal{H}^{(k)}(\Sigma) \to \mathcal{H}^{(k+1)}(\Sigma)$. If we wish to be careful, we can write $\nabla^{(k)}_z, \nabla^z_{(k)}$ to record the helicity space being acted on. To see how these operators act on tensor fields, we first calculate the Christoffel symbols for (15):

$$\Gamma^z_{zz} = \frac{1}{2}\gamma^{z\alpha}(2\partial\gamma_{z\alpha} - \partial_\alpha\gamma_{zz}) = \gamma^{z\bar{z}}\partial\gamma_{z\bar{z}} = 2\partial\omega, \quad \Gamma^{\bar{z}}_{\bar{z}\bar{z}} = 2\bar{\partial}\omega, \tag{17}$$

with other symbols (in particular $\Gamma^z_{\bar{z}z}$) vanishing. From the usual rules for covariant differentiation, it follows that for $T \in \mathcal{H}^{(k)}(\Sigma)$,

$$\nabla_z T^{z\cdots z}_{z\cdots z} = (\partial + k\Gamma^z_{zz})T^{z\cdots z}_{z\cdots z} = (\partial + 2k\partial\omega)T^{z\cdots z}_{z\cdots z} \tag{18}$$

$$\nabla^z T^{z\cdots z}_{z\cdots z} = \gamma^{z\bar{z}}(\bar{\partial} + k\Gamma^z_{\bar{z}z})T^{z\cdots z}_{z\cdots z} = \gamma^{z\bar{z}}\bar{\partial}T^{z\cdots z}_{z\cdots z}. \tag{19}$$

Define an inner product on $\mathcal{H}^{(k)}(\Sigma)$ by

$$(T, T') := \int d^2z\, \sqrt{\gamma}[\gamma_{z\bar{z}}]^k \bar{T}T'. \tag{20}$$

Here, $\bar{T}$ the complex conjugate of T, with all z indices replaced by $\bar{z}$ indices, and coefficients also conjugated. Our covariant derivatives are *adjoint* with respect to the inner product in the sense that if $T \in \mathcal{H}^{(k)}(\Sigma)$ and $T' \in \mathcal{H}^{k-1}(\Sigma)$, then a calculation [4] shows that

$$(T, \nabla^z T') = -\overline{(\nabla_z T, T')}. \tag{21}$$

The difference in kernels between "adjacent" operators is governed by the *Riemann-Roch theorem*:

$$\dim_{\mathbb{C}} \ker \nabla_z^{(k)} - \dim_{\mathbb{C}} \ker \nabla^z_{(k-1)} = (2k-1)(g-1). \tag{22}$$

This is proved in [5] using Faddeev-Popov ghosts, and the Atiyah-Singer index theorem in [4]. I have sadly omitted the proof for lack of space.

On each helicity subspace $\mathcal{H}^{(k)}(\Sigma)$, we can define a *Laplacian* by acting with our adjoint derivatives in turn. There are two choices depending on the order we take them, which we label $\Delta^+ := -\nabla_z\nabla^z$ or $\Delta^- := -\nabla^z\nabla_z$. These operators are different, as we can see from (18) and (19). For instance, recalling that $\gamma^{z\bar{z}} = 2e^{-2\omega}$,

$$\begin{aligned} \Delta^+_{(k)} &= -\nabla_z^{(k+1)}\nabla^z_{(k)} = 2\big(\partial + 2(k+1)\partial\omega\big) \cdot e^{-2\omega}\bar{\partial} \\ &= -2e^{-2\omega}[-2\partial\omega + \partial\bar{\partial} + 2(k+1)\partial\omega\bar{\partial}] \\ &= -2e^{-2\omega}[\partial\bar{\partial} + 2k(\partial\omega)\bar{\partial}]. \end{aligned} \tag{23}$$

A similar calculation shows that

$$\Delta^-_{(k)} = -2e^{-2\omega}[\partial\bar{\partial} + 2k(\partial\omega)\bar{\partial} + 2k(\partial\bar{\partial}\omega)]. \tag{24}$$

The spectrum of these Laplacians will prove crucial. Incidentally, heat kernel regularisation of these spectra provides *another* proof of Riemann-Roch [1].

3. String measure

3.1. **Groups and Jacobians.**

We now return to the loop amplitudes in (8). The basic strategy is sketched in (6) and (7): perform the integral over gauge orbits by integrating over the full, gauge-redundant space and dividing by the volume of orbits. First of all, we note that for general D,

$$\begin{aligned} \mathrm{vol}(G_g) &= \mathrm{vol}(\mathrm{MCG}) \cdot \mathrm{vol}(\mathrm{Diff}_0 \rtimes \mathrm{Weyl}) \\ &= \mathrm{vol}(\mathrm{MCG}) \cdot \frac{\mathrm{vol}(\mathrm{Diff}_0)\mathrm{vol}(\mathrm{Weyl})}{\mathrm{vol}(\mathrm{CKG})}, \end{aligned} \tag{25}$$

where $\mathrm{CKG}(\Sigma)$ is the overlap of Diff_0 and Weyl, i.e. the group of "small" conformal transformations in $\mathrm{Diff}_0(\Sigma)$. We can think of these as elements of G_0 which have no effect on the metric.

For the *critical dimension* $D = 26$, the combined measure $\mathcal{D}X\,\mathcal{D}\gamma$ is Weyl invariant,[2] so the gauge orbit is *larger* by a factor $\mathrm{vol}(\mathrm{CKG})$:

$$\mathrm{vol}(G_g) = \mathrm{vol}(\mathrm{MCG})\mathrm{vol}(\mathrm{Diff}_0)\mathrm{vol}(\mathrm{Weyl}), \quad \text{for } D = 26. \tag{26}$$

[2]Of course, *classically* the theory is conformally invariant for all D.

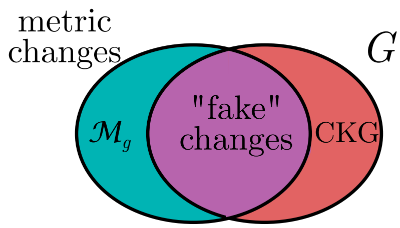

Figure 6. Following [7], a Venn diagram with physical changes to the metric (blue), gauge changes (purple), and gauge transformations which do not affect the metric (red).

Figure 7. (a) Two factorisations of the field space: $\mathrm{Met} \times \mathrm{Map}$ and $\mathrm{orbit} \times \mathrm{slice}$. The volume elements differ by a Jacobian factor $\mathcal{J}$. (b) The measure for small deformations mapped by MCG onto the whole space.

The vectors $\epsilon^\alpha \in \mathrm{CKG}(\Sigma)$ are called *conformal killing vectors* (CKVs), and satisfy $\delta\gamma_{\alpha\beta} = \gamma_{\alpha\beta}\nabla_\lambda\epsilon^\lambda$. Equivalently, they are in the kernel of the operator

$$[P_1\epsilon]_{\alpha\beta} := \frac{1}{2}\left(\nabla_\alpha\epsilon_\beta + \nabla_\beta\epsilon_\alpha - \gamma_{\alpha\beta}\nabla_\lambda\epsilon^\lambda\right). \tag{27}$$

This is just the projector onto symmetric traceless tensors. The space of CKVs is finite dimensional,[3] with basis vectors Φ_j^α and associated (real) deformation parameters a^j for $j = 1, \ldots, k$.

The adjoint $P_1^\dagger$ with respect to our inner product (20) maps symmetric traceless tensors to vectors, with the form

$$[P_1^\dagger h]_\alpha := -2\nabla^\beta h_{\alpha\beta}. \tag{28}$$

It is easiest to see that these are adjoint in complex coordinates. First, from definition (27), we have

$$[P_1\epsilon]^{zz} = \nabla^z_{(1)}\epsilon^z, \quad [P_1\epsilon]_{zz} = \nabla_z^{(1)}\epsilon_z. \tag{29}$$

Similarly, we apply (28) to find

$$[P_1^\dagger h]_z = \nabla_z^{(2)}h^{zz}, \quad [P_1^\dagger h]^z = \nabla^z_{(-2)}h_{zz}. \tag{30}$$

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[3] In complex coordinates, these are just entire holomorphic functions. On the plane, there are infinitely many of these, but on the sphere we only have global conformal transformations ($k = 2\cdot 3 = 6$). On the torus, only *constant* functions are entire, corresponding to translations ($k = 2$). Finally, for genus $g \geq 2$, entirety is so restrictive that *no* CKVs exist. See [5].

Since $\nabla_z^{(k+1)}$ and $\nabla^z_{(k)}$ are adjoint by (21), it follows that P_1 and $P_1^\dagger$ are adjoint. From (29) and (30), we can view P_1 and $P_1^\dagger$ as acting on $\mathcal{H}^{(1)} \oplus \mathcal{H}^{(-1)} \to \mathcal{H}^{(2)} \oplus \mathcal{H}^{(-2)}$ and $\mathcal{H}^{(2)} \oplus \mathcal{H}^{(-2)} \to \mathcal{H}^{(1)} \oplus \mathcal{H}^{(-1)}$ respectively. This means that the Laplacians of §2.2 appear in the product $P_1^\dagger P_1$:

(31)
$$P_1^\dagger P_1 = \begin{bmatrix} \nabla_z^{(2)} & \\ & \nabla^z_{(-2)} \end{bmatrix} \begin{bmatrix} \nabla^z_{(1)} & \\ & \nabla_z^{(-1)} \end{bmatrix} = \begin{bmatrix} \Delta^+_{(k)} & \\ & \Delta^-_{(k)} \end{bmatrix}.$$

To find the volumes in (25), we change from (X, γ) coordinates to coordinates along the moduli space (called the *gauge slice*) fibred by orbits:

$$\mathcal{D}X\,\mathcal{D}\gamma = \mathcal{J}\,\mathcal{D}[\text{orbit}] \cdot \mathcal{D}[\text{slice}].$$

The integral along $\mathcal{D}[\text{orbit}]$ cancels the group volumes, leaving an integral over moduli space with a Jacobian factor $\mathcal{J}$. We will use this trick to calculate $\text{vol}(\text{Diff}_0)\text{vol}(\text{Weyl})$, and deal with the remaining volumes in other ways. In particular, we will restrict the path integral to *small deformations* of the metric $\gamma_{\alpha\beta}$ and fields X^μ,

$$\mathcal{D}X\,\mathcal{D}\gamma \to \mathcal{D}[\delta X]\,\mathcal{D}[\delta\gamma].$$

The "large" components are obtained from the action of MCG.

To actually calculate these Jacobians, we need to specify the integration measure for each variable. In complex coordinates, we define the measure for any tensor field ϑ using the inner product (20):

(32)
$$1 := \int \mathcal{D}\vartheta\,\exp\left[-\frac{1}{2}||\vartheta||^2\right], \quad ||\vartheta||^2 := (\vartheta, \vartheta).$$

Small deformations of ϑ will lead to a Gaussian integral we can explicitly solve to find the Jacobian. This is like the elementary trick used to find the polar Jacobian $\mathcal{J} = r$:

$$1 = \int \frac{\mathrm{d}v_x\,\mathrm{d}v_y}{2\pi}\,e^{-(v_x^2+v_y^2)/2} = \mathcal{J}\int \frac{\mathrm{d}v_r\,\mathrm{d}v_\theta}{2\pi}\,e^{-(v_r^2+r^2v_\theta^2)/2} = \mathcal{J}r^{-1}.$$

3.2. **Metric deformations.** Let's consider small deformations of the metric. We can split these into three components: Weyl transformations (with parameter $\delta\omega$), diffeomorphisms (parameter ϵ^α), and (complex) displacements in Teichmüller space. We label the tangent vectors of $\text{Teich}(\Sigma)$ by μ_i for $i = 1, \ldots, n$, $n = \dim_{\mathbb{C}} \text{Teich}(\Sigma)$, accompanied by deformation parameters τ^i. The μ_i, called *Beltrami differentials*, are symmetric, traceless 2-tensors since they live in the tangent space of $\text{Teich}(\Sigma)$.

We can shift the contribution of CKVs into the Weyl component, and project ϵ onto *traceless* symmetric deformations $P_1\epsilon$, since the trace component is a CKV. Alternatively, we can choose $\tilde{\epsilon} \in (\ker \nabla_z)^\perp$. Similarly, we keep only the orthogonal component of μ_i, $\mu_i^\perp \in \ker P_1^\dagger = (\text{im}\,P_1)^\perp$. Morally, the result is that we can decompose the set of metric deformations into the orthogonal sum

$$\{\delta\gamma\} = \text{Weyl} \oplus \text{im}\,P_1 \oplus \ker P_1^\dagger.$$

Let's see to the details. In complex coordinates the metric deformation reads

(33)
$$\delta\gamma_{zz} = \nabla_z\tilde{\epsilon}_z + \tau^i\mu_{izz}, \quad \delta\gamma_{z\bar{z}} = \delta\omega\gamma_{z\bar{z}}.$$

Split μ_i over the image of P_1 and the kernel of $P_1^\dagger$:

$$\mu_i := P_1 v_i + \alpha_i^\ell \phi_\ell, \tag{34}$$

where v_i is a vector chosen for each μ_i, and ϕ_ℓ is a basis of $\ker P_1^\dagger$. We can incorporate the first part into $\tilde{\epsilon}$ by defining $\bar{\epsilon} := \tilde{\epsilon} + 2\tau^i v_i$. The $\phi_i \in \ker P_1^\dagger$ are called *quadratic differentials*. Taking an inner product of (34) with ϕ_m, we find

$$\alpha_i^\ell = [(\phi,\phi)^{-1}]^{\ell m}(\mu_i, \phi_m). \tag{35}$$

where we regard $[(\phi,\phi)]_{\ell m} := (\phi_\ell, \phi_m)$ as a matrix.

By orthogonality and (35), we have

$$\begin{aligned} ||\delta\gamma||^2 &= ||\delta\gamma_{z\bar{z}}||^2 + ||\nabla_z \bar{\epsilon}_z||^2 + \bar{\tau}^i \tau^j (\alpha_i^\ell \phi_\ell, \alpha_j^m \phi_m) \\ &= ||\delta\omega||^2 + ||\nabla_z \bar{\epsilon}_z||^2 + \bar{\tau}^i \tau^j [(\phi,\phi)^{-1}]^{\ell m} (\phi_\ell, \mu_i)(\mu_j, \phi_m). \end{aligned} \tag{36}$$

In particular, we are using that $\gamma_{zz}, \gamma_{z\bar{z}}$ live in orthogonal helicity spaces. Using (32) and the Gaussian function integral (with $\mathcal{O}$ an operator on tensors ϑ)

$$\int \mathcal{D}\vartheta \, \exp\left[-\frac{1}{2}\vartheta\mathcal{O}\vartheta\right] = C_\vartheta [\det \mathcal{O}]^{-1/2},$$

we can calculate the Jacobian for the metric deformation:

$$\begin{aligned} 1 &= \int \mathcal{D}\gamma \, \exp\left[-\frac{1}{2}||\delta\gamma||^2\right] \\ &= \mathcal{J}_\gamma \int \mathrm{d}^n\tau \, \mathcal{D}\bar{\epsilon} \, \mathcal{D}\phi \, \exp\left[-\frac{1}{2}\left(||\nabla_z\bar{\epsilon}_z||^2 + ||\tau^i \phi_{izz}||^2 + ||\delta\omega\gamma_{z\bar{z}}||\right)\right] \\ &= \mathcal{J}_\gamma \int \mathcal{D}[\delta\omega] \, \exp\left[-\frac{1}{8}e^{4\omega}||\delta\omega||^2\right] \int \mathcal{D}\bar{\epsilon} \, \exp\left[-\frac{1}{8}||P_1\bar{\epsilon}||^2\right] \\ &\qquad \times \int \mathrm{d}^n\tau \, \exp\left[-\frac{1}{2}\left(\bar{\tau}^i\tau^j[(\phi,\phi)^{-1}]^{\ell m}(\phi_\ell,\mu_i)(\mu_j,\phi_m)\right)\right] \\ &= C_\omega C_{\bar{\epsilon}} C_\tau \mathcal{J}_\gamma \left[\frac{\det|\phi,\mu|^2 \det{}'(P_1^\dagger P_1)}{\det(\phi,\phi)}\right]. \end{aligned}$$

Note that since τ is complex, we do the Gaussian functional integral twice. Throwing away the irrelevant (divergent) constant $C_\omega C_{\bar{\epsilon}} C_\tau$ and using (31), we find

$$\mathcal{J}_\gamma = \left[\frac{\det|\phi,\mu|^2 \det{}'(\Delta^+\Delta^-)}{\det(\phi,\phi)}\right]. \tag{37}$$

Note that we have also shifted $\epsilon \to \bar{\epsilon}$. We can shift back and calculate the associated measure in the same way. The details are similar to our derivation of (37), and we find

$$\begin{aligned} 1 &= \int \mathcal{D}\epsilon \, \exp\left[-\frac{1}{2}||\epsilon||^2\right] \\ &= \mathcal{J}_\epsilon \int \mathrm{d}^n\tau \, \mathrm{d}^k a \, \mathcal{D}\bar{\epsilon} \, \exp\left[-\frac{1}{2}||\bar{\epsilon}||^2 - \frac{1}{2}||a^j\Phi_j||^2 - 2||\tau^i v_i||^2\right] \\ &\sim \mathcal{J}_\epsilon \left[\det(\Phi,\Phi)\right]^{-1/2}. \end{aligned} \tag{38}$$

We end this section by noting that the kernel of $P_1 = \nabla^z_{(1)} \oplus \nabla_z^{(-1)}$ is CKG, while the kernel of $P_1^\dagger = \nabla_z^{(2)} \oplus \nabla^z_{(-2)}$ gives a basis of quadratic differentials for Teichmüller space [3]. Thus, the Riemann-Roch theorem (22) implies

$$\begin{aligned}\dim_{\mathbb{C}} \ker P_1^\dagger - \dim_{\mathbb{C}} \ker P_1 &= \dim_{\mathbb{C}} \ker \nabla_z^{(2)} - \dim_{\mathbb{C}} \ker \nabla^z_{(1)} \\ &\quad - \dim_{\mathbb{C}} \ker \nabla_z^{(-1)} + \dim_{\mathbb{C}} \ker \nabla^z_{(-2)} \\ &= 3(g-1) \\ &= \dim \mathrm{Teich} - \dim \mathrm{CKVs}.\end{aligned} \tag{39}$$

Since there are no CKVs for $g \geq 2$, we recover the result quoted earlier for the dimension of Teichmüller (and hence moduli) space:

$$\dim_{\mathbb{R}}(\mathrm{Teich}(\Sigma_g)) = \dim_{\mathbb{R}}(\mathcal{M}_g) = -3\chi(g).$$

3.3. **Field deformations.** We can carefully perform the deformations δX in slice-orbit coordinates, but there is a simpler way to do things. Recall that the fields live in the space $X \in \mathrm{Map}(\Sigma_g, M)$, with *coset representatives* $\tilde{X} \in \mathcal{E}_g$ on each orbit. A deformation of X will decompose into a deformation of the representative $\tilde{X}$,[4] and Taylor expansion terms proportional to ϵ and $a^j\Phi_j$.

Since (37) and (38) *already* rotate into ϵ, we can ignore that deformation and restrict to the change of coodinates $X \to (\tilde{X}, a)$,[5] with associated Jacobian $\mathcal{J}_X$. As it turns out, we can reduce $\mathcal{J}_X\, e^{-S_{\mathcal{P}}^X}$ to an integral over the *redundant* field space X, since

$$\int \mathcal{D}X\; e^{-S_{\mathcal{P}}^X} = \int \mathrm{d}^k a\, \mathcal{D}X\, \mathcal{J}_X\, e^{-S_{\mathrm{P}}^X} = \mathrm{vol(CKG)} \int \mathcal{D}\tilde{X}\, \mathcal{J}_X\, e^{-S_{\mathcal{P}}^X}, \tag{40}$$

where we used the fact that the Jacobian are independent of the CKVs.

We can explicitly perform the functional integral over X. First, note that we can integrate the X component of the Polyakov action by parts, yielding

$$S_{\mathrm{P}}^X[X,\gamma] = -\frac{T}{2}\int d^2\sigma\, \sqrt{\gamma}\, X^\mu \Delta X_\mu = -\frac{T}{2}(X, \Delta X)_{\mathrm{LV}}, \tag{41}$$

where Δ is the usual scalar Laplacian, and $(\cdot,\cdot)_{\mathrm{LV}}$ is the natural inner product for a multiplet of worldsheet scalars which act as a spacetime Lorentz vector:

$$\Delta := -\frac{1}{\sqrt{\gamma}}\partial_\alpha \sqrt{\alpha}\gamma^{\alpha\beta}\partial_\beta = \nabla^\alpha\partial_\alpha, \quad (X,Y) := \int d^2\sigma\, \sqrt{\gamma}\, X^\mu Y_\mu. \tag{42}$$

Let's expand the X^μ in an eigenbasis $\{(\psi_n, \lambda_n)\}$ of Δ on Σ_g, with $\lambda_n \geq 0$ since Δ is positive definite. We will assume our target spacetime has critical dimension $D = 26$. Separating out the zero mode ψ_0, we find

$$X^\mu = \sum_{n\geq 0} C_n^\mu \psi_n := X_0^\mu + \sum_{n \geq 1} C_n^\mu \psi_n. \tag{43}$$

[4]This does not appear in the Jacobian, since it contributes $\int \mathcal{D}[\delta\tilde{X}]\exp(-||\delta\tilde{X}||^2) = 1$.

[5]The ϵ variation here will be an off-diagonal "coupling" between the metric and field deformations. This means it doesn't contribute to the final Gaussian determinant, and we can ignore it! See [4] for details.

The C_n^μ parameterise the path integral. By orthonormality of the $\{\psi_n\}$, and performing a Gaussian integral on the second last line, the $\mathcal{D}X$ path integral in (40) becomes

$$\int \mathcal{D}X\ \exp^{-\frac{1}{2}T(X,\Delta X)} = \int \prod_{n\mu} \mathrm{d}C_n^\mu\, e^{-\frac{T}{2}\lambda_n C_n^2} = \frac{1}{\psi_0^{26}}\left(\int \mathrm{d}X_0\right)^{26} \prod_{n\geq 1}\left(\frac{2\pi}{T\lambda_n}\right)^{26/2} \sim \frac{(\det'\Delta)^{-13}}{\psi_0^{26}}.$$

Here, we have thrown away a divergent constant, and $\det'$ denotes the determinant excluding the zero mode. Let $\mathcal{A}[\gamma]$ denote the area of the worldsheet, so that the zero mode has *normalised* value $\psi_0 := \sqrt{\mathcal{A}[\gamma]}$. Thus, (44) becomes

$$\int \mathcal{D}X\, e^{-S_{\mathrm{P}}} \sim \left[\frac{\det'\Delta}{\mathcal{A}[\gamma]}\right]^{13}. \tag{44}$$

3.4. **Final result.** We can finally assemble our results. Using (26), (37), (38), (40) and (44), we can rewrite the loop amplitudes in (8) as

$$\begin{aligned} Z_g &= \frac{g_{\mathrm{s}}^{2-2g}}{\mathrm{vol}(G_g)} \int \mathrm{d}^n\tau\, \mathcal{D}\epsilon\, \mathcal{D}[\delta\omega]\, \mathcal{J}_\gamma \mathcal{J}_\epsilon \int \mathcal{D}\tilde{X}\, \mathcal{J}_X\, e^{-S_{\mathcal{P}}^X} \\ &= \frac{g_{\mathrm{s}}^{2-2g}}{\mathrm{vol(MCG)vol(CKG)vol(Diff_0)vol(Weyl)}} \int \mathrm{d}^n\tau\, \mathcal{D}\epsilon\, \mathcal{D}[\delta\omega]\, \mathcal{J}_\gamma \mathcal{J}_\epsilon \int \mathcal{D}X\, e^{-S_{\mathcal{P}}^X} \\ &= \frac{g_{\mathrm{s}}^{2-2g}}{\mathrm{vol(MCG)vol(CKG)}} \int_{\mathrm{Teich}(\Sigma_g)} \mathrm{d}^n\tau \left[\frac{\det|\phi,\mu|^2 \det(\Phi,\Phi)\det{}'(\Delta^+\Delta^-)}{\det(\phi,\phi)}\right]\left[\frac{\det'\Delta}{\mathcal{A}[\gamma]}\right]^{13}, \end{aligned} \tag{45}$$

where we cancelled the volume of $\mathrm{Weyl}(\Sigma_g)$ by integrating over $\delta\omega$, and the volume of $\mathrm{Diff}_0(\Sigma_g)$ by integrating over ϵ.

We are left with a *finite-dimensional* integral over Teichmüller space. Dividing by the volume of the mapping class group just yields the moduli space. For $g \geq 2$, there are no CKVs, so (45) simplifies to

$$Z_g = g_{\mathrm{s}}^{2-2g} \int_{\mathcal{M}_g} \left[\mathrm{d}^n\tau\, \frac{\det|\phi,\mu|^2}{\det(\phi,\phi)}\right] \det{}'(\Delta^+\Delta^-) \left[\frac{\det'\Delta}{\mathcal{A}[\gamma]}\right]^{13}.$$

The first term in square brackets is a Kähler form on the moduli space, known as the *Weil-Petersson measure* ω_{WP} [3].[6] It's possible to show that the spectrum of Δ^+ and Δ^- agree [4], so we have the slightly tidier answer for $g \geq 2$:

$$\boxed{Z_g = g_{\mathrm{s}}^{2-2g} \int_{\mathcal{M}_g} \omega_{\mathrm{WP}}\ \det{}'(\Delta^\pm)^2 \left[\frac{\det'\Delta}{\mathcal{A}[\gamma]}\right]^{13}.} \tag{46}$$

Physically, (46) tells us that in order for *nothing to happen*, string theory must probe the symplectic structure of every compact Riemann surface Σ_g and sound out its harmonics via $\Delta^\pm$ and Δ. To paraphrase Mark Kac, the loop amplitude Z_g hears the shape of the Riemann surface Σ_g.

[6] Technically, we need to invoke the *Uniformisation Theorem* to first map Σ_g to a canonical constant-curvature surface Σ_g^{UC} [4]. In the same way the torus is obtained by quotienting the complex plane by a lattice $T^2 = \mathbb{C}/\Lambda$, higher genus constant-curvature spaces are obtained by quotienting the upper-half plane by a subgroup $\Gamma_g \subset \mathrm{SL}(2,\mathbb{R})$, $\Sigma_g^{\mathrm{UC}} = \mathbb{H}/\Gamma_g$.

3.5. **Extensions.** We have merely dipped our toes into the deep waters of string perturbation theory. Even confining ourselves to closed bosonic strings, there are many directions we can swim. A first step is to generalise (46) to include vertex insertions. The order $g \geq 2$ contribution to the scattering amplitude (6) is easily found [3]:

$$\mathcal{A}_g^{(b)}(\{\alpha\}) = g_{\mathrm{s}}^{2-2g} \int_{\mathcal{M}_g} \omega_{\mathrm{WP}}\ \det{}'(\Delta^{\pm})^2 \left[\frac{\det' \Delta}{\mathcal{A}[\gamma]}\right]^{13} \left\langle \prod_\alpha V_\alpha \right\rangle_X,$$

where $\langle\cdot\rangle_X$ denotes the path integral with respect to $\mathcal{D}X$ only.

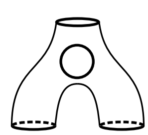
Another question is whether we can write (46) explicitly. One approach is to evaluate the spectra of the Laplacians $\Delta, \Delta^{\pm}$ using *Ray-Singer analytic torsion* [7]. It's also possible to use algebraic rather spectral invariants. Reference [2] proves that (46) can be written $Z_g = g_{\mathrm{s}}^{\chi(g)} \int_{\mathcal{M}_g} |\mu_g|^2$, where μ_g is an object from algebraic geometry called the *Mumford form* on Σ_g.

There is more than on way to skin a cat, or in this case, perform a path integral. First, we can derive Jacobians using Faddeev-Popov ghosts, and the corresponding BRST symmetry makes the physics of loop amplitudes clearer [5]. Second, we can use combinatorial techniques to sew higher genus Riemann surfaces Σ_g out of spheres, with associated patching for the moduli space and CFTs. This lets us prove *unitarity* of string scattering [5]. It also makes the analogy to Feynman diagrams particularly manifest, since cut-and-paste constructions are represented as graphs.

This concludes our brief tour of string amplitudes. We finish with a quote (a twist on Kronecker attributed to Lipman Bers):

> *God, if She exists, created the natural numbers* $\{0, 1, 2, \ldots\}$*, and compact Riemann surfaces. The rest of mathematics is man-made.*

String theory is tied in a deep way to these "God-given" objects. Perhaps one day, viewed from the right perspective, it will seem just as natural.



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