UBC Physics Circle
The Penrose Singularity Theorem

David Wakeham

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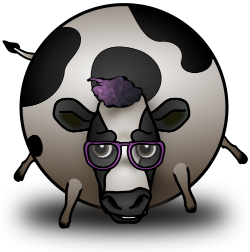
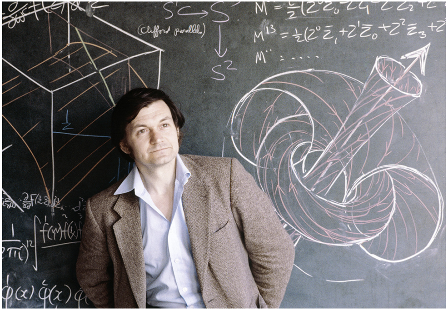


Overview

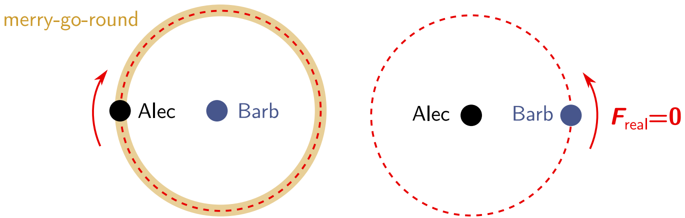
- The 2020 Nobel Prize in Physics was awarded to Andrea Ghez and Reinhard Genzel and Sir Roger Penrose.



- Penrose showed mathematically that singularities generally form inside black holes. We'll understand why!

Newton's second law

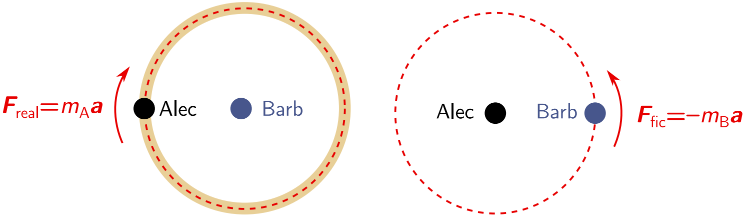
- Recall Newton's second law, $F = ma$. A force (F) accelerates an object (a), resisting with some inertia (m).
- Forces don't depend on how we describe them. Right? **Smart Alec** and **Sharp Barb** will guide us through.
- **Alec** sits on a merry-go-round, while **Barb** is stationary.



- **Smart Alec**: "Barb accelerates without force!"

Fictitious forces

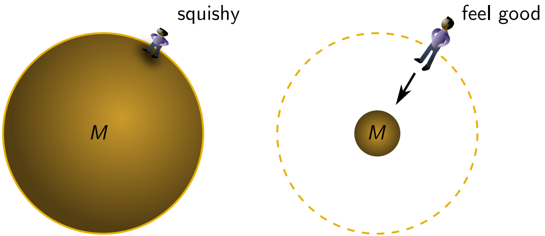
- **Sharp Barb**: "You're in an accelerating frame."
- **Alec**: "How do I tell?" **Barb**: "Use an accelerometer!"



- **Alec**: "If I don't have one?" **Barb**: "Look out for fictitious forces proportional to my mass."
  If Barb's mass m_B increases, $\mathbf{F}_{\text{fic}}$ increases accordingly.

Gravity is fake

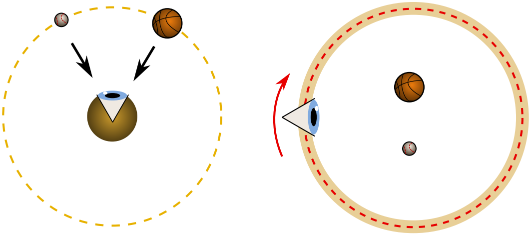
- **A**: "That can't be right. Weight is proportional to mass, $W = mg$. But gravity isn't fictitious!"
- **B**: "Standing on the ground, you feel gravity's pull, just like the merry-go-round. Maybe it is fake!"



- **A**: "And the inertial frames, where forces are real?"
  B: "The ones where you feel no gravity: freefall!"

The equivalence principle

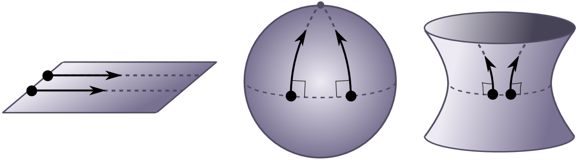
- Albert Einstein had the same insight as Alec and Barb in 1911. He called it the equivalence principle, since it explains the equivalence of inertial and gravitational mass.
- Put simply, baseballs and basketballs fall the same way.



- This is just like a stationary baseball and basketball rotating the same way, viewed from a merry-go-round!

Gravity is curvature

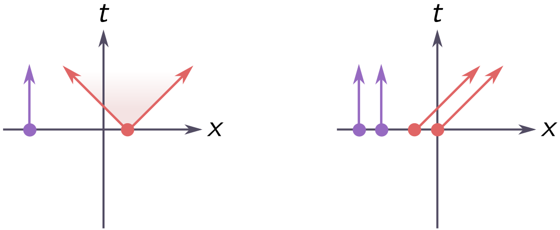
- **A**: “Hang on a sec: things fall to the ground and orbit. Gravitation happens! How can this be if gravity is fake?”
- **B**: “Gravity is fake but space is curved.”
- On the plane, parallel, straight lines stay parallel forever.



- Curvature occurs when straight lines start parallel but change their mind. These lines are called geodesics: they are locally straight. They look straight when you zoom in!

Straight lines in spacetime

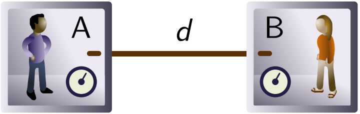
- **A**: "I understand how curvature can bring straight lines together. But what about stationary objects?"
- **B**: "Nothing stays still if you add a time coordinate!"
- We draw a plane with x and t axes. Stationary objects are purple. Light rays are red, and border the light cone.



- In flat spacetime, parallel lines remain that way. In curved spacetime, geodesics can change their mind.

Spacetime curvature

- Alec and Barb set up nearby labs, connected by a taut string and with synchronized onboard clocks, time τ.



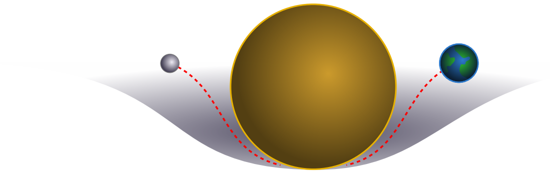
- The labs start at fixed distance d and free fall. With string and clocks, they measure the acceleration a of d.
- The spacetime curvature R is defined by

$$a = -Rd.$$

In flat space, $R = 0$. Positive R means they get closer together. Negative R means they drift apart.

General relativity

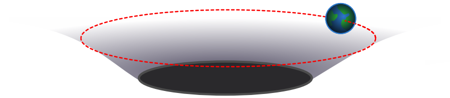
- **A**: "So we can measure curvature. But what causes it?"
- **B**: "Newton told us mass creates a gravitational force. If that force is fake, then mass must create curvature."



- Matter tells space how to curve.
  Space tells matter how to move. (John Wheeler)
- Einstein's field equations (1915) translate this into math. We could learn more about these equations . . .

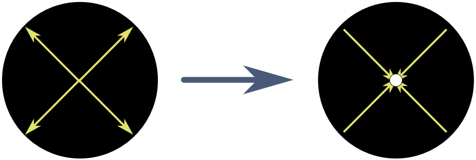
Schwarzschild's surprise

- Instead, we'll do something easier: earn a Nobel prize!
- In 1915, Karl Schwarzschild worked out how spherical mass curves spacetime. You get orbits and so on.
- But for a very dense sphere, there are two surprises:
  - there is a light-trapping region;
  - there is a singularity, where spacetime breaks down.



A singular inconvenience

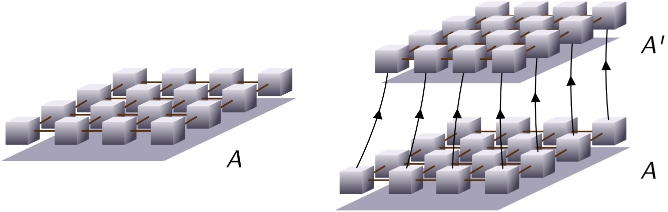
- For our purposes, a singularity occurs when you cannot head into the future. There is literally nowhere to go!
- **A**: "Maybe singularities are just a bug in Schwarzschild's code. In a more general black hole, with less symmetry and filled with matter, there may be no singularity."



- **B**: "Einstein agreed — he didn't like that his theory predicted its own demise! But in 1965, Penrose showed that trapping light leads inevitably to singularities. "

A congruence of labs

- We start by considering a network of nearby labs , called a congruence, spanning an area A.

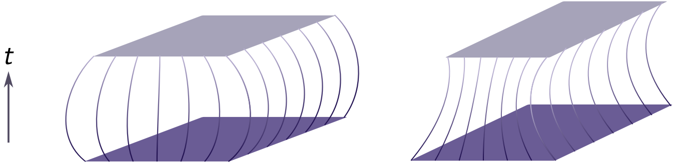


- The area spanned can change with time. The expansion is the fractional rate of change

$$\theta = \frac{1}{A}\frac{\Delta A}{\Delta \tau}.$$

The focusing theorem

- The expansion itself can change, with some rate of change we call bending, $\alpha = \Delta\theta/\Delta\tau$.
- The sign of α tells us if the labs converge or diverge.



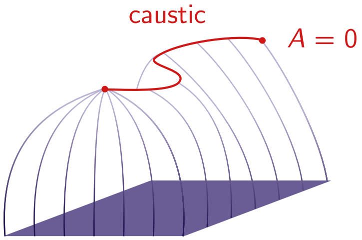
$\alpha < 0$ (converging) $\alpha > 0$ (diverging)

- Einstein's equations (with ordinary matter) imply that gravity is attractive: labs in freefall converge. More precisely, the focusing theorem states

$$\alpha \leq -\frac{\theta^2}{3}.$$

Caustics and benefits

- The focusing theorem has an important consequence.
- If θ is ever negative, the network collapses to zero area in finite time. The labs collide! This is called a caustic.

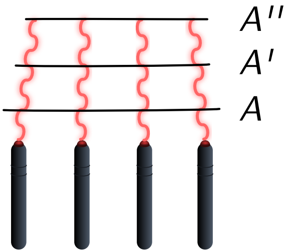


- Idea: use the focusing theorem to show that $|\theta|$ undergoes runaway growth, with $\theta = -\infty$ in finite time. See the exercises for more details!

From labs to laser pointers

- Everything we've said so far about focusing holds for a congruence of laser pulses, with two differences.

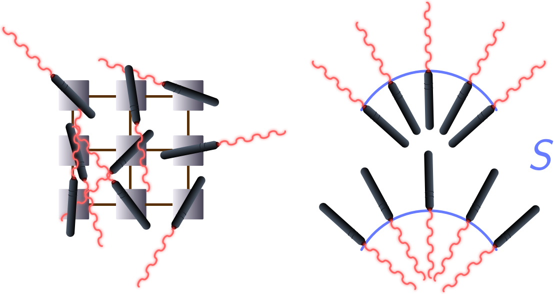
1. Clocks attached to light rays stop due to time dilation.



We calculate expansion θ and bending α with respect to number of wavelengths, instead of onboard clock time τ.

Surface grids

2. Light rays have to move. If we fire a ball of laser pointers, rays will intersect, however we orient them.

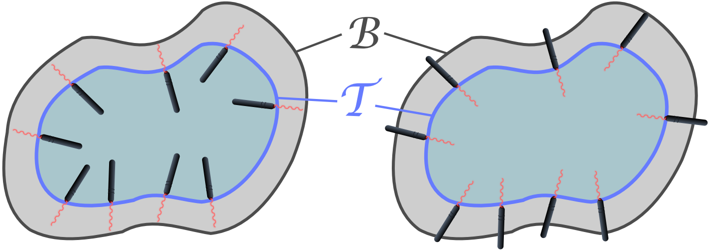


Instead, we choose a surface S (and side) to fire from.

- The focusing theorem holds for surface grids of lasers. The pulses are called a null geodesic congruence.

Trapped null surfaces

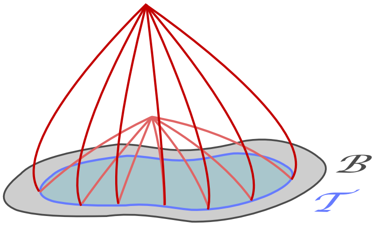
- Let $\mathcal{B}$ be a region of space which traps light. Select a closed surface $\mathcal{T}$ just inside $\mathcal{B}$. There are two surface grids on $\mathcal{T}$: outward-directed and inward-directed.



- The area spanned by inward bound light rays shrinks. But because $\mathcal{B}$ traps light, the outward bound rays shrink too!
- Such a $\mathcal{T}$ is called a trapped null surface (TNS).

Colliding lasers

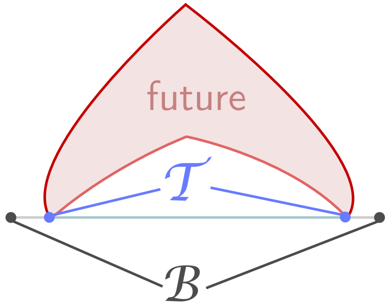
- Light rays in both directions from $\mathcal{T}$ have $\theta < 0$.
- The focusing theorem implies that caustics develop in a finite number of wavelengths. Thus, every outward- or inward-bound light ray leaving $\mathcal{T}$ collides with another.



- It turns out that these caustics indicate the presence of a singularity. There is no future for $\mathcal{T}$ beyond them!

The blob at the end of time

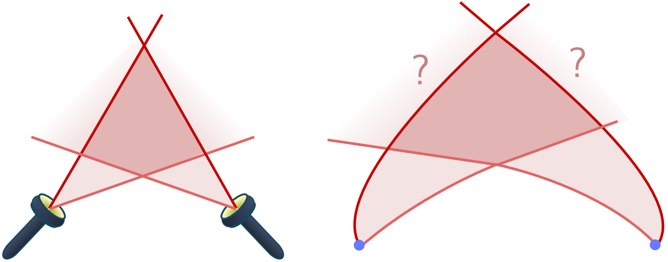
- Let's make sure we understand why the future ends. It's easier to see if we take a two-dimensional slice.
- Both black hole and closed trapped surface are two dots.



- We claim the future of $\mathcal{T}$ is the finite blob above.

The flashlight future

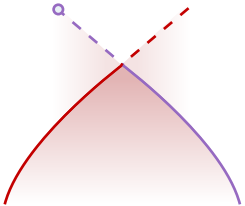
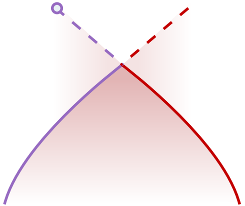
- **A**: " Just because light rays intersect doesn't mean there is a problem. Consider two crossed flashlights!



- The flashlights don't end time. Can't you extend the light rays from the blob, and get more future?"

Zig-zag light rays

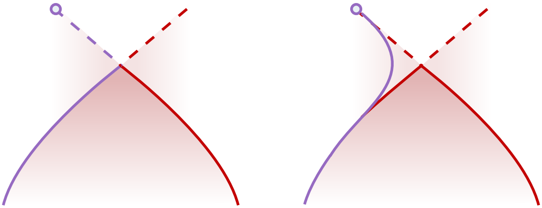
- **B**: "Suppose we can. Now pick a point on one of the extended rays. How can a light ray get there?"

- **A**: "Just follow the original ray."
- **B**: "Right. But because we have extended through the caustic, a zig-zag light ray path can also get there."

A shortcut for rockets

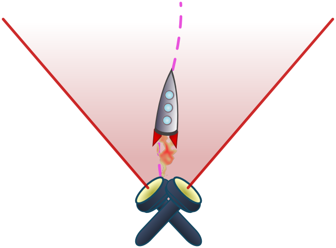
- **A**: "There is no law against zig-zags."
- **B**: "What about shortcuts? I can deform the zig-zag into a path inside the blob. A fast rocket can also get there!



- In fact, the same argument shows a rocket can get to any point on a ray extended beyond a caustic."

The edge of the future

- **B**: "A light ray is quicker than a rocket. So the edge of the future is always reached by light rays, not rockets."



- **A**: "I get it! Points on any extended ray can be reached by rockets, so they cannot be on the edge of $\mathcal{T}$'s future."
- **B**: "Exactly. And since there are no other candidate light rays to bound it, the future must end in the blob!"

Conclusion

- This proves the Penrose singularity theorem: a surface where outgoing light rays shrink destroys the future.
- We know black holes exist (see the Nobel for Genzel and Ghez!) and black holes create trapped null surfaces.
- Thus, general relativity predicts singularities inside!

Thanks for listening!