UBC Physics Circle: The Penrose Singularity Theorem

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Problems

1. **Focusing in one dimension.** Let's apply the focusing theorem to Alec and Barb's labs, when there is only a single spatial dimension. In this case, instead of area A we have the distance d, and the fractional rate of change is

$$\theta = \frac{v}{d},$$

where v is the rate of change of the length of the string. Using calculus, it is possible to show that

$$\alpha = \frac{ad - v^2}{d^2}.$$

(a) From the focusing theorem for labs, deduce that

$$a \leq \frac{2v^2}{3d}.$$

(b) Now recall that we defined curvature R was defined by $a = -Rd$, where the labs are initially paralell. Using the previous question, show that the focusing theorem for labs implies $R \geq 0$, i.e. the 2D universe is flat or positively curved.

2. **Caustics in finite time.** In this problem, we we will show that caustics develop in finite time from negative expansion, assuming the focusing theorem. Although we will state the problem in terms of τ, the time for clocks on board a lab, the same math holds when we replace time with λ, the number of wavelengths of a laser pulse.

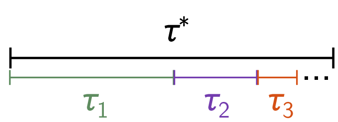

Figure 1: Splitting $\tau^* = 6/|\theta_0|$ into a geometric series.

Suppose that a congruence has negative expansion $\theta_0 = -|\theta_0| < 0$. We will give a rough argument that the area vanishes in time $\tau^* = 6/|\theta_0|$. Our trick will be to split τ^* into a *geometric series*:

$$\tau^* = \frac{\tau^*}{2} + \frac{\tau^*}{4} + \frac{\tau^*}{8} + \cdots = \tau_1 + \tau_2 + \tau_3 + \cdots ,$$

where we define $\tau_n = \tau^*/2^n$.

(a) The focusing theorem states that $\alpha \leq -\theta^2/3$. Explain why setting these to be equal gives an upper bound on the time until the caustic.

(b) We will set $\alpha = -\theta^2/3$ to find an upper bound on the time it takes to form a caustic. We proceed in the time steps defined above. Argue that, approximately, after time $\tau_1 = \tau^*/2 = 3/|\theta_0|$, the expansion is

$$\theta_1 = 2\theta_0.$$

(c) Similarly, by computing a few examples (or any other method), explain why after n time steps,

$$\theta_n = 2^n\theta_0.$$

(d) Argue that $\theta \to -\infty$ in a finite time, and hence $A = 0$ at or before $\tau^* = 6/|\theta_0|$.

(e) *Bonus*. Using calculus, show that caustics occur at or before $\tau^*_{\text{calc}} = 3/|\theta_0|$.

3. **Apparent vs event horizons.** In th e talk, I defined a black hole region as one which contains null trapped surfaces $\mathcal{T}$. This is actually quite a strong notion of the black hole region, called the *apparent horizon* because it's "apparent": you can *tell* when you're inside using a grid of laser pointers! More precisely, an apparent horizon $\mathcal{B}$ has the property that any closed surface $\mathcal{T}$ strictly within it is a trapped null surface.

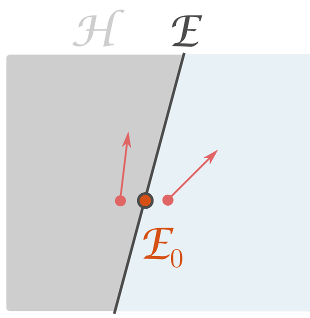

Figure 2: The black hole and event horizon.

But there is a sneakier notion of horizon. Suppose somebody notices they are falling into a black hole and sends a distress signal saying "Help! I'm falling into a black hole!" We will ignore whether or not we can save them: the question is, can someone far away from the black hole receive the message at all? The *(global) black hole* $\mathcal{H}$ is the whole region of spacetime from which *you cannot even send a distress signal*. The boundary of this region is call the *event horizon* $\mathcal{E}$.

(a) Give an informal argument that $\mathcal{E}$ is described by a null geodesic congruence, i.e. is generated by laser pulses. *Hint*. You might want to consider a slice of $\mathcal{E}_0$ at some fixed time, and the outward-directed surface grid.

(b) Explain why an apparent horizon $\mathcal{B}$ will always lie at or inside the event horizon $\mathcal{E}$.

4. **The area theorem.** Our last problem will use the techniques we've developed to give a rough proof of the famous *area theorem*, due to Stephen Hawking in 1971. It states that black holes always get bigger. Let's see how it goes!

(a) In the last question, we showed that the event horizon of a black hole can be viewed as a null congruence. It therefore has some expansion θ. If $\theta < 0$, argue that the black hole will collapse to a singularity in a finite number of wavelengths.

(b) If this singularity forms, it is *naked*, i.e. not clothed by a black hole horizon, since the horizon itself has collapsed! The *cosmic censorship hypothesis*, conjectured by Penrose in 1969, is that Nature forbids naked singularities. Assuming this conjecture, show that $\theta \geq 0$ for event horizons.

(c) Conclude that the area of a black hole can never decrease.

In fact, there is nothing in our argument that restricts us to a *single* black hole, since the event horizon could have different components. The same reasoning therefore shows that the total area of *any number* of black holes will increase!

Hawking noted that this was rather odd. In physics, most quantities can either increase or decrease; they do not track the flow of time. The famous exception is *entropy*, the amount of disorder, or a little less mysteriously, the number of ways a system can be, which increases as time goes on, at least for thermodynamic systems with many particles. This suggests that the area of a black hole is connected to its entropy! This observation lead to the remarkable discovery that black holes themselves obey all four laws of thermodynamics, once suitable analogues to ordinary thermodynamic quantities have been identified. Sadly, we must leave the topic of black hole thermodynamics for another time!

Solutions

1. (a) The focusing theorem states that $\alpha \leq -\theta^2/3$, or

$$\alpha = \frac{ad - v^2}{d^2} \leq -\frac{v^2}{3d^2} \quad \Longrightarrow ad \leq \frac{2v^2}{3}.$$

   (b) Since the labs are parallel, we set $v = 0$. The previous question shows $ad \leq 0$, and hence $-Rd^2 \leq 0$. Since $d^2 > 0$ is strictly positive, we have $R \geq 0$.

2. (a) The more negative α is, the more it bends the geodesics towards each other, and hence the quicker the caustic arrives. Setting $\alpha = -\theta^2/3$ therefore gives the *slowest* approach to a caustic.

   (b) By definition,

$$\alpha = \frac{\Delta\theta}{\Delta\tau} \quad \Longrightarrow \quad \theta_1 = \theta_0 + \alpha\Delta\tau.$$

   After the first time step, this gives

$$\theta_1 = \theta_0 + \alpha\tau_1 = \theta_0 - \frac{\theta_0^2}{3} \cdot \frac{3}{|\theta_0|} = 2\theta_0.$$

   (c) Let's do another example. We can do the same thing as the last question to get

$$\theta_2 = \theta_1 + \alpha\tau_2 = \theta_1 - \frac{\theta_1^2}{3} \cdot \frac{3}{2\theta_0} = \theta_1 - \frac{\theta_1^2}{3} \cdot \frac{3}{\theta_1} = 2\theta_1.$$

   Looks like a pattern! It's easy to see that the pattern continues, simply doubling θ at each point.

   Bonus. If you like math, there is a proof by induction. Suppose that $\theta_n = 2^n\theta_0$. Then

$$\theta_{n+1} = \theta_n + \alpha\tau_{n+1} = \theta_n - \frac{\theta_n^2}{3} \cdot \frac{6}{2^{n+1}|\theta_0|} = \theta_n + \frac{\theta_n^2}{3} \cdot \frac{3}{\theta_n} = 2\theta_n.$$

   (d) We're taking something negative and doubling it an infinite number of times, so we will certainly get $-\infty$. This happens in a *finite* time τ^* since we have obtained our steps by splitting that up! Finally, the only way

$$\theta = \frac{1}{A}\frac{\Delta A}{\Delta\tau}$$

   can blow up is if $A = 0$.

   (e) *This requires some knowledge of differentiation.* Really, $\alpha = d\theta/d\tau$ is the derivative of θ. If we set $\alpha = -\theta^2/3$, we have a slightly confusing looking equation to solve:

$$\frac{d\theta}{d\tau} = -\frac{\theta^2}{3}.$$

   Let's invert both sides:

$$\frac{d\tau}{d\theta} = -\frac{3}{\theta^2}.$$

Now we view τ as a function of θ, and ask what has the derivative $-3/\theta^2$. If you have a bit of calculus under your belt, you can antidifferentiate immediately to find

$$\tau = \frac{3}{\theta} + c = \frac{3}{\theta} - \frac{3}{\theta_0},$$

where we determine the constant c from the fact that $\theta = \theta_0$ at $\tau = 0$. Now, caustics occur when $\theta = -\infty$. We will call the corresponding time $\tau = \tau^*_{\text{calc}}$. In this case, the first term vanishes, and we are left with

$$\tau^*_{\text{calc}} = -\frac{3}{\theta_0} = \frac{3}{|\theta_0|}.$$

3. (a) Let's follow the hint and consider a slice $\mathcal{E}_0$. If we set up a grid of lasers on the surface and shoot outward pulses, what will happen? Well, venture a little ways outside the horizon and the messages will reach the region very distant from the black hole. Venture a little inside and they will be trapped inside. So right on the event horizon itself, the light gets stuck! It simply hovers there. This means that the *horizon $\mathcal{E}$ itself* is threaded by this outgoing null congruence.

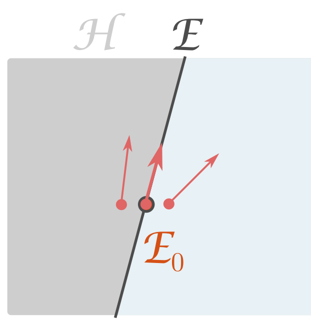

Figure 3: The outgoing light rays from $\mathcal{E}_0$ run along the horizon itself.

   (b) If $\mathcal{T}$ is a trapped null surface, then the network of outgoing light rays gets smaller. There is no way it can reach distant regions outside the black hole! By definition, any surface within an apparent horizon $\mathcal{B}$ is a trapped null surface. These trapped surfaces can get arbitrarily close to the apparent horizon, suggesting that we cannot send a message to the distant region from $\mathcal{B}$ itself. Otherwise, we could shrink the surface by a teensy weensy bit, to get $\mathcal{B}'$, and send a message to the distant region — but this is impossible since $\mathcal{B}'$ is trapped!

4. (a) From the last problem, we know that the event horizon $\mathcal{E}$ can be viewed as the outgoing lasers from a surface grid. As discussed in the talk, these obey the focusing theorem, and if $\theta < 0$ at any point, then by the same reasoning, the horizon itself will develop caustics and hence a singularity after a finite number of wavelengths.

   (b) There is no problem with the horizon collapsing, except that the singularity would be there for everyone to see. So, if the cosmic censorship conjecture is true, this is not possible, and we must have $\theta \geq 0$ at all times on the horizon.

   (c) Since θ is the (fractional) change in area, $\theta \geq 0$ means the area can never decrease!