UBC Virtual Physics Circle
A Hacker's Guide to Random Walks

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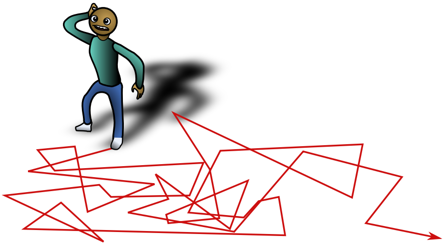
Overview

- Today, we're going to learn about random walks.

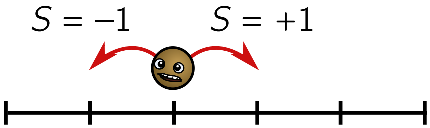


- This is the motion executed by a drunkard!
  But also polymers, photons in the sun, atoms...
- We will take an elementary approach.

All the math!

Random walks: steps

- A random walk consists of random steps S.
  This could be in one or more dimensions.



- The sum of N steps is

$$T_N = S_1 + \cdots + S_N.$$

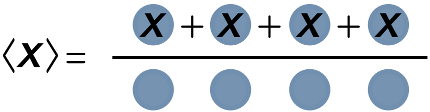
We would like to understand some aspects of T_N.

Basic probability: averages

- We're going to need a few basic facts about probability.
- First of all, suppose X is a random number (or function of random numbers). The average $\langle X \rangle$ is

$$\langle X \rangle = \frac{\text{sum of results for } X \text{ over many experiments}}{\text{number of experiments}}.$$

- We won't need probability, just averages! In pictures:

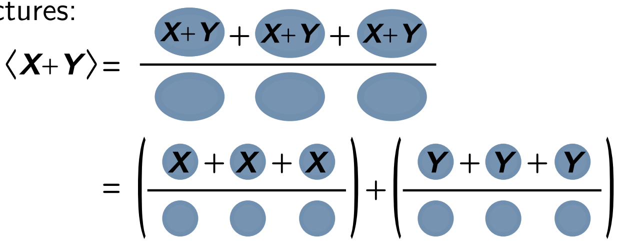


Basic probability: sum rule

- **Sum rule.** If X and Y are random, then

$$\langle X+Y\rangle = \frac{\text{sum of } (X+Y)}{\text{number of experiments}} = \frac{(\text{sum of } X) + (\text{sum of } Y)}{\text{number of experiments}} = \langle X\rangle + \langle Y\rangle.$$

In pictures:



Random walks: unbias

- **Unbiased.** We say the steps are unbiased if $\langle S \rangle = 0$.
- It follows from the sum rule that T_N is unbiased:

$$\langle T_N \rangle = \langle S_1 \rangle + \cdots + \langle S_N \rangle = 0.$$

  Random walks go nowhere on average! Boring.
- Let a drunkard move back or forward a step by tossing a fair coin S. In N tosses, we get $\sim N/2$ tails andheads, so

$$\langle S \rangle = \frac{N/2 - N/2}{bcN} = 0.$$

  On average, the drunkard remains where they are!

Basic probability: uncorrelation

- **Uncorrelation.** We say that X and Y are uncorrelated if

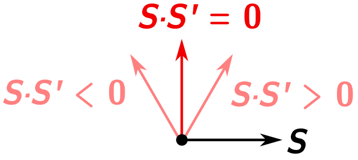
$$\langle XY \rangle = \langle X \rangle \langle Y \rangle.$$

If they are unbiased, then uncorrelation means $\langle XY \rangle = 0$.

- Unbiased random vectors $\vec{S}, \vec{S}'$ are uncorrelated if

$$\langle \vec{S} \cdot \vec{S}' \rangle = 0,$$

where $\vec{S} \cdot \vec{S}' = 0$ if they are perpendicular.



Random walks: deviation

- Consider a walk of N unbiased, uncorrelated steps:

$$\vec{T}_N = \vec{S}_1 + \vec{S}_2 + \cdots + \vec{S}_N.$$

  We know that the average $\langle \vec{T}_N \rangle = 0$ is boring.

- A better measure is the standard deviation, $\sqrt{\langle \vec{T}_N^2 \rangle}$, measuring the size of the region covered by the walk.
- Note that $(x+y)^2 = x^2 + y^2 + 2xy$ generalizes to

$$\begin{aligned}\vec{T}_N^2 &= (\vec{S}_1 + \cdots + \vec{S}_N)^2 \\ &= \vec{S}_1^2 + \cdots + \vec{S}_N^2 + 2(\vec{S}_1 \cdot \vec{S}_2 + \cdots + \vec{S}_{N-1} \cdot \vec{S}_N),\end{aligned}$$

Random walks: finale!

- Now we just take averages of $\vec{T}_N^2$ using the sum rule.
- If steps are unbiased/uncorrelated, the cross-terms vanish:

$$\langle \vec{T}_N^2 \rangle = \langle \vec{S}_1^2 \rangle + \cdots + \langle \vec{S}_N^2 \rangle.$$

- If each step length is ℓ, then $\langle \vec{S}_1^2 \rangle = \ell^2$. Then

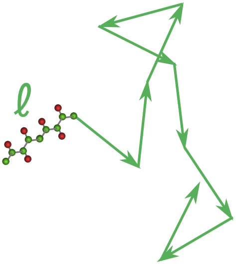
$$d = \sqrt{\langle \vec{T}_N^2 \rangle} = \sqrt{\ell^2 + \cdots + \ell^2 \rangle} = \sqrt{N}\ell.$$

- This is our big result: a random walk tends to spread a distance $\propto \sqrt{N}$, where N is the number of steps.

Applications

Polymers: intro

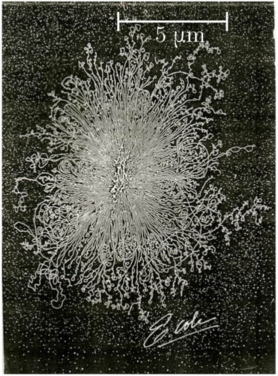
- Our first application is to long molecules called polymers.
- A polymer is a chain of approximately straight links of length ℓ. These links can form a random walk in space.



- The most famous polymer is DNA. It is not usually a random walk — unless it spills out of the nucleus!

Polymers: E. Coli genome

- **Exercise 1.** Below is the spilled DNA of an E. coli bacterium. A rigid chunk has length $\ell = 48$ nm, corresponding to ~ 140 base pairs (bp).



- Estimate the total length L of the genome in bp.

Polymers: E. Coli genome

- **Solution.** From the scale, we have $d \sim 5\,\mu\text{m}$. Using $d \sim \sqrt{n}\ell$, the total number of links is

$$n \sim \frac{d^2}{\ell^2} = \left(\frac{5 \times 10^{-6}}{48 \times 10^{-9}}\right)^2 \approx 11 \times 10^3.$$

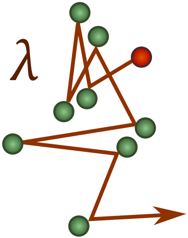
- Multiplying by the number of base pairs in a chunk gives

$$L = (11 \times 10^3)(140 \text{ bp}) \sim 1.5 \text{ Mbp}.$$

- Biologists tell us the correct answer is $L = 4.9$ Mbp. We're within an order of magnitude! (Physics dance.)

Collisions: intro

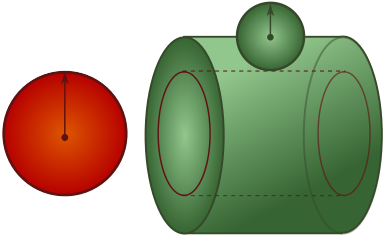
- Collisions are another rich source of random walks.
- In many situations, particles move in straight lines until they collide! This resets their direction randomly.



- This looks like a random walk, with step length set by something called the mean free path (mfp) λ.

Collisions: cylinders

- To find the mfp, we'll use collision cylinders. This is the volume a particle sweeps out as it moves.
- A useful tweak is to choose a volume such collisions occur when the centre of another particle lies inside.



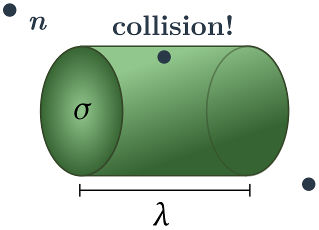
- **Exercise 2.** A sphere of radius R collides with spheres of radius r. Show the collision cylinder has radius $R + r$.

Collisions: density and mfp

- The cylinder scattering cross-section is σ. Move a distance d, and the collision cylinder has volume $V = \sigma d$.
- If there are n particles per unit volume, then

$$Vn = \sigma dn = 1 \quad \Longrightarrow \quad d = \frac{1}{\sigma n}.$$

- You expect a collision after a distance $d = 1/\sigma n$. But this is just the mfp! So $\lambda = 1/\sigma n$.

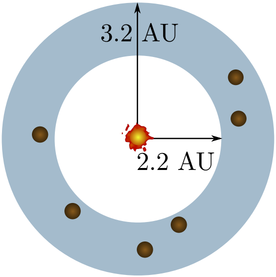


Asteroid belt

- Our first application is asteroids!
- The asteroid belt is ring between Jupiter and Mars, 2.2 to 3.2 astronomical units (AU) from the sun, where

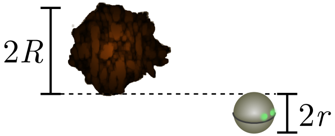
$$1 \text{ AU} = 1.5 \times 10^8 \text{ km}.$$

- We never program space probes to avoid asteroids. Why?



Asteroid belt

- The belt has 25M asteroids, average diameter 10 km.
- **Exercise 3.** (a) What is the density of asteroids, *n*?
- (b) Space probes are much smaller than asteroids. Explain why the collision "strip" has width $\sigma \approx 10$ km.



- (c) Find the mean free path of a space probe. Conclude it almost never collides with asteroids!

Asteroid belt

- **Solution.** (a) Density is total number divided by area:

$$n = \frac{25 \times 10^6}{\pi(3.2^2 - 2.2^2)(1.5 \times 10^8 \text{ km})^2} \approx 7 \times 10^{-11} \text{ km}^{-2}.$$

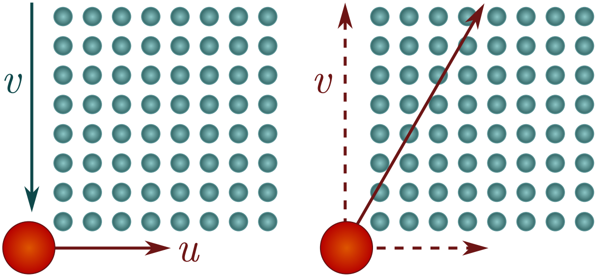
- (b) Approximate the space probe as a point. It collides with an asteroid when it's less than an asteroid radius away! So the collision width $\sigma \approx 10$ km.
- (c) Using our formula for mean free path,

$$\lambda = \frac{1}{\lambda\sigma} \approx \frac{1}{10(7 \times 10^{-11})} \text{ km} \approx 10 \text{ AU}.$$

This is much bigger than the width of the asteroid belt!

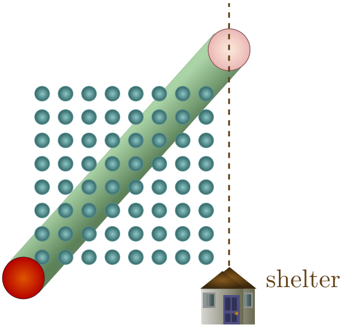
Running in the rain

- Another application is the age-old (Vancouver-relevant) question: should you walk or run in the rain?
- We ignored the motion of the asteroids...
- But rain is clearly moving! We deal with this by doing everything in the reference frame of the rain.



Running in the rain

- Suppose shelter is some distance d away. In the rain frame, it moves up at the same speed as you.
- We (naturally) model people as spheres of radius R.



- We should minimise the length of our collision cylinder.

Running in the rain

- **Exercise 4.** (a) If you run at speed u, raindrops have density n and speed v, argue you collide with k drops for

$$k = nd\sigma\sqrt{1 + (v/u)^2} = nd\pi R^2\sqrt{1 + (v/u)^2}.$$

  This decreases as we make u bigger!
- (b) If wind blows the rain towards the shelter, argue there is a finite optimal speed to run.
- Bonus. If rain blows towards shelter with horizontal speed u' and falls at speed v, show the optimal speed is v^2/u'.

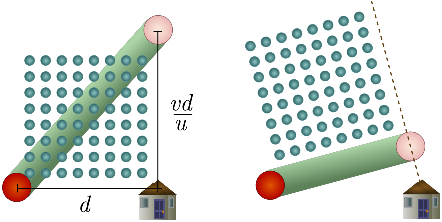
Running in the rain

- **Solutions.** (a) It takes time $t = d/u$ to reach shelter. In that time, you travel up $tv = vd/u$ in the rain frame. So

$$\text{total distance} = \sqrt{d^2 + (vd/u)^2} = d\sqrt{1 + (v/u)^2}.$$

We then multiply by cross-section $\sigma = \pi R^2$ and density n.

- (b) The optimal collision cylinder is shown right:



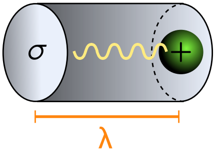
- This corresponds to a finite horizontal speed.

A walk in the sun

- Let's finish by adding random walks back into the mix.
- In the sun, photons are constantly colliding with hydrogen nuclei. The cross-section and density of nuclei are

$$\sigma = 6 \times 10^{-29} \text{ m}^2, \quad n = 5 \times 10^{32} \text{ m}^{-3}.$$

- **Exercise 5.** What is the mean free path of a photon?

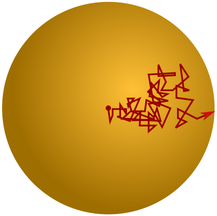


- **Solution.** From $\lambda = 1/\sigma n$, we have

$$\lambda = [(6 \times 10^{-29})(5 \times 10^{32})]^{-1} \text{ m} \approx 3 \times 10^{-5} \text{ m}.$$

A walk in the sun

- The sun has a radius of $R_{\odot} = 7 \times 10^8$ m and photons travel at $c = 3 \times 10^8$ m/s between collisions.
- **Exercise 6.** If a photon starts in the centre, roughly how long does it take to random walk out of the sun?



- Remember that spread obeys $d \sim \sqrt{N}\lambda$.

A walk in the sun

- **Solution.** First, we relate time t to number of steps N:

$$c = \frac{\text{total length of path}}{t} = \frac{N\lambda}{t} \quad \Longrightarrow \quad N = \frac{ct}{\lambda}.$$

If the photon spreads out a distance $d \sim R_\odot$, our law of random walks states $R_\odot \sim \sqrt{N}\lambda$. Hence

$$\begin{aligned} N = \frac{ct}{\lambda} &\sim \frac{R_\odot^2}{\lambda^2} \\ \Longrightarrow \quad t &\sim \frac{R_\odot^2}{c\lambda} = \frac{(7 \times 10^8 \text{ m})^2}{(3 \times 10^8 \text{ m/s})(3 \times 10^{-5} \text{ m})} \\ &= 5.4 \times 10^{13} \text{ s}. \end{aligned}$$

This is about 2 million years!

Questions?

Next time: Einstein's atomic escapades!