PHYC20014 Physical Systems

Wave Theory and Fourier Analysis: Assignment 3

Due Friday, October 21, 2016 at 5:00 pm

1. Surf's up! You are on holiday in a sun-kissed Californian beach town. Each morning, shoobies, locals and kooks flock to the water to surf the perfect A-frames rolling into the bay. The waves can be modelled as an initial value problem in 1D:

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \frac{\partial^2 \psi}{\partial x^2}, \quad \psi(x,0) = f(x), \quad \dot{\psi}(x,0) = g(x). \tag{1}$$

Before you go surfing, you decide to do a little Fourier analysis. As usual, F and G denote the Fourier transforms of f and g respectively.

(a) Fourier transform (1) with respect to x only, i.e.

$$\psi(x,t) \to \Psi(u,t).$$

Simplify the resulting equation using the derivative theorem for Fourier transforms. You should find

$$\ddot{\Psi}(u,t) = -(2\pi uv)^2 \Psi(u,t), \quad \Psi(u,0) = F(u), \quad \dot{\Psi}(u,0) = G(u). \tag{2}$$

(b) For each value of u, (2) is a second-order ODE with respect to t. Solve this set of ODEs to find

$$\Psi(u,t) = F(u)\cos(2\pi uvt) + \frac{1}{2\pi uv} G(u)\sin(2\pi uvt).$$

(c) By taking the inverse Fourier transform, reproduce d'Alembert's formula:

$$\psi(x,t) = \frac{1}{2}\big[f(x-vt) + f(x+vt)\big] + \frac{1}{2v}\int_{x-vt}^{x+vt} g(y)\,dy.$$

After drawing equations on the sand for a while, the onshore wind picks up and the waves start to get "blown out", i.e. damped. They now satisfy a modified wave equation

$$\frac{\partial^2 \psi}{\partial t^2} + \frac{2}{\tau}\frac{\partial \psi}{\partial t} = v^2 \frac{\partial^2 \psi}{\partial x^2}, \tag{3}$$

where τ is the damping constant and we ignore initial conditions.

(d) Take the Fourier transform of (3). Deduce that Ψ obeys

$$\ddot{\Psi}(u,t) + \frac{2}{\tau}\dot{\Psi}(u,t) + (2\pi uv)^2 \Psi(u,t) = 0. \tag{4}$$

(e) Substitute the power-law ansatz $\Psi = h(u)e^{\lambda t}$ into (4), where $h(u) \neq 0$. Argue that, at fixed u, the system only exhibits wave-like (oscillatory) behaviour for

$$|u| > \frac{1}{2\pi v\tau}.$$

Since $|u|$ is the spatial frequency of a wave, when the wind blows waves below the cutoff frequency do not propagate.

Due to the wind, the surf is "mushy" and unrideable; you have nothing to do except sit on the beach and watch the ocean. After careful observation, you realise that waves in shallow water obey the *nonlinear* PDE

$$\frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi}{\partial x} + \frac{\partial^3 \psi}{\partial x^3} = 0. \tag{5}$$

(f) From (5), derive the corresponding integro-differential equation for Ψ,

$$\dot{\Psi}(u,t) + (2\pi i u)^3 \Psi(u,t) + 2\pi i \int_{-\infty}^{\infty} \xi \Psi(\xi,t)\Psi(u-\xi,t)\, d\xi = 0.$$

This doesn't make life easier! To solve (5), there is a nonlinear version of the Fourier transform called the *inverse scattering transform*, but it is way beyond the scope of this modest assignment question.

(g) It turns out that (5) has the solitary wave solution[1]

$$\psi(x,t) = \frac{\omega - 4k^3}{k} + 12k^2 \operatorname{sech}^2(kx - \omega t + \delta),$$

where k, ω, δ are arbitrary. Draw a picture of the wave and describe explicitly how k, ω, δ control its shape and speed.

(h) From the physical requirement $\lim_{x \to \pm\infty} \psi(x,t) = 0$, relate k and ω. Hence, show that the height of the wave is proportional to the speed.

[2 + 1 + 4 + 1 + 3 + 2 + 3 + 2 = 18 marks]

[1] You may verify this for your own peace of mind, but I suggest using Mathematica.

PHYC20014 Physical Systems

Fourier Analysis and Applications: Assignment 3

Solutions

---

1. Surf's Up!

(a) The wave equation is

$$\frac{\partial^2\psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}.$$

Fourier transforming with respect to x and using $\hat{\mathcal{F}}[f'] = 2\pi iu\hat{\mathcal{F}}[f]$, we obtain

$$(2\pi iu)^2\Psi = \frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2}. \quad [1]$$

This is the result we want. Fourier transforming the initial conditions gives $\Psi(u,0) = \hat{\mathcal{F}}[\psi(x,0)](u) = \hat{\mathcal{F}}[f](u) = F(u)$ and similarly $\dot{\Psi}(u,0) = \hat{\mathcal{F}}[g](u) = G(u)$. [1]

(b) For each fixed value of u, we have a simple ODE to solve: the harmonic oscillator! The general solution is just a linear combination of sine and cosine terms with frequency $\omega = 2\pi uv$, and solving for the initial conditions gives

$$\Psi(u,t) = F(u)\cos(2\pi uvt) + \frac{1}{2\pi uv}G(u)\sin(2\pi uvt). \quad [1]$$

(c) We apply $\hat{\mathcal{F}}$ to both sides, liberally using properties from Tutorial 3. The LHS is $\hat{\mathcal{F}}[\Psi] = \hat{\mathcal{F}}^2[\psi] = \psi(-x,t)$. The first term on the RHS is (using the convolution theorem and the Fourier transform of cos [1])

$$\begin{aligned}
\hat{\mathcal{F}}[F(u)\cos(2\pi uvt)](-x) &= (f_- * \hat{\mathcal{F}}[\cos(2\pi uvt)])(-x,t) \\
&= \frac{1}{2}\int_{-\infty}^{\infty} f(-\xi)\Big[\delta(x - vt - \xi) + \delta(x + vt - \xi)\Big]\,d\xi \\
&= \frac{1}{2}\big[f(x - vt) + f(x + vt)\big]. \quad [1]
\end{aligned}$$

This is the first term in d'Alembert's formula; so far so good. From the derivative theorem,

$$\hat{\mathcal{F}}\left[\frac{1}{2\pi uv}G(u)\right] = \frac{i}{v}\int^{-x} g(y)\,dy. \quad [1]$$

By reasoning similar to the above, we then obtain

$$\begin{aligned}
\hat{\mathcal{F}}\left[\frac{1}{2\pi u}G(u)\sin(2\pi uvt)\right](x,t) &= \left(\hat{\mathcal{F}}\left[\frac{1}{2\pi u}G(u)\right] * \hat{\mathcal{F}}\big[\sin(2\pi uvt)\big]\right)(x,t) \\
&= \frac{1}{2v}\int_{x-vt}^{x+vt} g(y)\,dy. \quad [1]
\end{aligned}$$

(d) This is the same as part (a), except for the middle term. Thus, we get

$$\ddot{\Psi}(u,t) + \frac{2}{\tau}\dot{\Psi}(u,t) + (2\pi i u)^2\Psi(u,t) = 0. \quad [1]$$

(e) Substituting $\Psi = e^{\lambda t}$ into (4) yields

$$\left[\lambda^2 + \frac{2}{\tau}\lambda + (2\pi u v)^2\right]\Psi(u,t) = 0. \quad [1]$$

Since $\Psi \neq 0$, λ must be a root of the quadratic in brackets. Oscillatory behaviour occurs when λ has an imaginary component [1], i.e. the discriminant of the quadratic is negative:

$$\Delta = \frac{4}{\tau^2} - 4(2\pi u v)^2 < 0 \quad \Longrightarrow \quad |u| > \frac{1}{2\pi v\tau}. \quad [1]$$

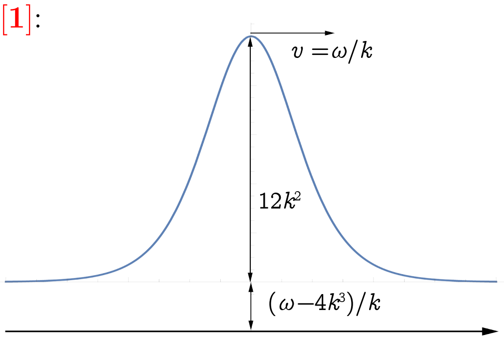
(f) We can deal with x derivatives using the derivative theorem. Now we also need the convolution theorem in the form:

$$\hat{\mathcal{F}}[fg] = F * G. \quad [1]$$

Fourier transforming both sides, we get

$$\begin{aligned} 0 &= \dot{\Psi}(u,t) + (2\pi i u)^3\Psi(u,t) + (\Psi * (2\pi i u)\Psi) \\ &= \dot{\Psi}(u,t) + (2\pi i u)^3\Psi(u,t) + 2\pi i\int_{-\infty}^{\infty}\xi\Psi(\xi,t)\Psi(u-\xi,t)\,d\xi. \qquad [1] \end{aligned}$$

(g) We first draw the wave [1]:



The dependence on $kx - \omega t$ shows that the wave travels to the right at speed $v \equiv \omega/k$, with initial phase offset δ. [1] It is shifted up or down by the constant term $(\omega - 4k^3)/k$ and has amplitude $H \equiv 12k^2$. [1]

(h) Since $\operatorname{sech}(y) \to 0$ as $y \to \pm\infty$,

$$\lim_{x\to\pm\infty}\psi(x,t) = \frac{\omega - 4k^3}{k}.$$

This vanishes provided $\omega = 4k^3$. [1] As we calculated in the previous question, the speed is $v = \omega/k = 4k^2$. Finally, the height of the wave is $H = 12k^2$, so the speed and height are related by

$$H = 3v. \quad [1]$$