PHYC20014 Physical Systems

Fourier Analysis and Optics: Assignment 2

Due —, October —, — at 5:00 pm

1. Willy Wien's selection machine. Suppose a positron (mass m and charge e) moves through an electromagnetic field with scalar potential $\phi(\mathbf{x}, t)$ and vector potential $\mathbf{A}(\mathbf{x}, t)$. As you know from your EM class, this means the electric and magnetic fields can be written

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi, \quad \mathbf{B} = \nabla \times \mathbf{A}.$$

The Lagrangian for the positron is

$$L(t, \mathbf{x}, \dot{\mathbf{x}}) = \frac{1}{2} m \dot{x}_j \dot{x}_j - e\left(\phi - \dot{x}_j A_j\right), \tag{1}$$

where x_i is the ith Cartesian coordinate and we sum over j (Einstein summation convention).

(a) Show that the conjugate momentum associated with coordinate x_i is

$$p_i = m\dot{x}_i + eA_i. \tag{2}$$

(b) When is p_i conserved?

(c) Convert L into the Hamiltonian for the system. You should find

$$H(t, \mathbf{x}, \mathbf{p}) = \frac{1}{2m}\left(p_j - eA_j\right)^2 + e\phi. \tag{3}$$

(d) Express Hamilton's equations for (3) in the following form:

$$\dot{x}_i = \frac{1}{m}\left(p_i - eA_i\right) \tag{4}$$

$$\dot{p}_i = \frac{e}{m}\left(p_j - eA_j\right)\frac{\partial A_j}{\partial x_i} - e\frac{\partial \phi}{\partial x_i}. \tag{5}$$

(e) In a *Wien filter*, the potentials are

$$\phi = Ex, \quad \mathbf{A} = (-By, 0, 0)$$

where E and B are constants. Without calculating anything, explain why p_z is conserved.

(f) Specialise Hamilton's equations to the filter. You should find

$$\dot{x} = \frac{1}{m}\left(p_x + Bey\right), \quad \dot{p}_x = -Ee, \quad \dot{p}_y = -Be\dot{x}, \quad \dot{y} = \frac{p_y}{m}.$$

(g) This arrangement acts as a velocity selector: a positron initially travelling in the positive y direction will be deflected unless it has a specific velocity. Show that the "magic" velocity is

$$v = \frac{E}{B}.$$

(h) What happens when $E > cB$?

[1 + 1 + 2 + 3 + 1 + 3 + 3 + 1 = 15 marks]

2. Fishy sums and hot donuts. *Poisson summation* is a deep relationship between Fourier series and Fourier transforms. We can exploit this to learn about hot donuts!

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be a real function, and define a new function

$$h(x) \equiv \sum_{n=-\infty}^{\infty} f(x+n). \tag{6}$$

Check that $h(x)$ has period $T = 1$.

(b) Assuming $h(x)$ satisfies the Dirichlet conditions, calculate the coefficients of the exponential Fourier series

$$h(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}.$$

You should find

$$c_n = F(n), \tag{7}$$

where $F \equiv \hat{\mathcal{F}}[f]$ is the Fourier transform of f. Deduce the *Poisson summation formula*,

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} F(n) e^{2\pi i n x}. \tag{8}$$

This relates the periodic sums of a function and its Fourier transform.

(c) Consider heat flow on an infinite, 1D wire. The temperature $T(x,t)$ obeys the *diffusion equation*,

$$\frac{\partial T}{\partial t} = D \frac{\partial^2 T}{\partial x^2}. \tag{9}$$

Suppose we start with a point-like spike, $T(x,0) = \delta(x)$. Show that the function

$$T(x,t) \equiv \frac{1}{\sqrt{4\pi D t}} e^{-x^2/4Dt}$$

solves (9) with initial condition $T(x,0) = \delta(x)$.[1] This is called the *heat kernel*. It is just the Green's function with the sign flipped.

[1] For checking the delta property, you are encouraged to use results from lectures.

(d) Now wrap the wire into a circle of unit circumference C. *Without* doing any calculations, argue that

$$S(\theta, t) = \frac{1}{\sqrt{4\pi D t}} \sum_{n=-\infty}^{\infty} e^{-(\theta+n)^2/4Dt} \tag{10}$$

is the heat kernel on the circle.[2] HINT. Remember that the periodic version of $\delta(x)$ is the Dirac comb $\text{Ш}_T(\theta)$ from Tutorial 1.

(e) Use Poisson summation to rewrite (10) as

$$S(\theta, t) = \sum_{n=-\infty}^{\infty} e^{-(2\pi n)^2 Dt} e^{2\pi i n\theta}. \tag{11}$$

(f) We can modify the initial temperature distribution to $S_g(\theta, 0) = g(\theta)$, where g has the exponential Fourier series

$$g(\theta) = \sum_{n=-\infty}^{\infty} d_n e^{2\pi i n\theta}.$$

Using (11) and the method of Green's functions (or otherwise), derive the identity

$$S_g(\theta, t) = \sum_{n=-\infty}^{\infty} d_n e^{-(2\pi n)^2 Dt} e^{2\pi i n\theta}.$$

(g) Mathematically, you can define a *donut* as the product of two circles, $C \times C$. The diffusion equation (9) becomes

$$\frac{\partial K}{\partial t} = D\left(\frac{\partial^2 K}{\partial \theta_1^2} + \frac{\partial^2 K}{\partial \theta_2^2}\right). \tag{12}$$

Verify that the heat kernel on the donut is

$$K(\theta_1, \theta_2, t) = \sum_{m,n=-\infty}^{\infty} e^{-4\pi^2(m^2+n^2)Dt} e^{2\pi i(\theta_1 m + \theta_2 n)}.$$

(h) **Bonus.** Heat flow on higher dimensional donuts is considered a trade secret by litigious, higher-dimensional donut vendors. Suppose we have an N-dimensional donut

$$\overbrace{C \times \cdots \times C}^{N \text{ times}}$$

with heat equation

$$\frac{\partial K}{\partial t} = D \sum_{i=1}^{N} \frac{\partial^2 K}{\partial \theta_i^2}.$$

Generalise the answer from (f) to find the heat kernel. Don't tell the vendors!

[2 + 4 + 4 + 2 + 3 + 2 + 3 + (2) = 20 marks]

[2] The heat equation on the circle is identical to (9), with θ replacing x.

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Solutions

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1. Willy Wien's selection machine.

(a) The conjugate momentum is

$$p_i \equiv \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i + eA_i. \quad [\mathbf{1}]$$

(b) The conjugate momentum p_i is conserved when x_i is cyclic, i.e. there is no x_i dependence in L. [**1**]

(c) We obtain the Hamiltonian by Legendre transformation [**1**]:

$$\begin{aligned} H(t, \mathbf{x}, \mathbf{p}) &= \dot{x}_j p_j - L \\ &= \dot{x}_j(m\dot{x}_j + eA_j) - \frac{1}{2}m\dot{x}_j\dot{x}_j + e\,(\phi - \dot{x}_j A_j) \\ &= \frac{1}{2}m\dot{x}_j\dot{x}_j + e\phi \\ &= \frac{1}{2m}\,(p_j - eA_j)^2 + e\phi \qquad [\mathbf{1}] \end{aligned}$$

using (2) on the first and last line.

(d) Hamilton's equations are

$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x_i}. \quad [\mathbf{1}]$$

Using (3), Hamilton's equations become

$$\begin{aligned} \dot{x}_i &= \frac{\partial H}{\partial p_i} = \frac{1}{m}\,(p_j - eA_j)\,\frac{\partial p_j}{\partial p_i} = \frac{1}{m}\,(p_i - eA_i) \qquad [\mathbf{1}] \\ \dot{p}_i &= -\frac{\partial H}{\partial x_i} = \frac{e}{m}\,(p_j - eA_j)\,\frac{\partial A_j}{\partial x_i} - e\frac{\partial \phi}{\partial x_i}. \qquad [\mathbf{1}] \end{aligned}$$

(e) There is no z dependence, so p_z is conserved. [**1**]

(f) Equations (4) and (5) become

$$\begin{aligned} \dot{x} &= \frac{1}{m}\,(p_x + Bey) \\ \dot{p}_x &= \frac{e}{m}\,(p_j - eA_j)\,\frac{\partial A_j}{\partial x} - e\frac{\partial \phi}{\partial x} = -Ee \\ \dot{p}_y &= \frac{e}{m}\,(p_j - eA_j)\,\frac{\partial A_j}{\partial y} - e\frac{\partial \phi}{\partial y} = -\frac{Be}{m}\,(p_x + Bey) \\ \dot{y} &= \frac{p_y}{m}, \quad \dot{z} = \frac{p_z}{m}, \quad \dot{p}_z = 0. \qquad [\mathbf{3}] \end{aligned}$$

(g) From the equations of motion, we see that setting $p_x = -Bey$ automatically leads to $\dot{x} = 0$, i.e. no deflection in the x direction. [1] To satisfy the equation for $\dot{p}_x$, we must have

$$-Ee = -eB\dot{y} = -eBv \quad \Longrightarrow \quad v = \frac{E}{B}. \quad [2]$$

The remaining equations are satisfied for $p_y = mv$ and $p_z = \text{const}$.

(h) In this case, the magic velocity exceeds the speed of light, $v = E/B > c$. Since nothing can travel faster than c, the positron must be deflected. [1]

2. Fishy sums and hot donuts.

(a) The function $h(x)$ has period $T = 1$ just in case $h(x+1) = h(x)$ for any x. [1] We now check if this holds:

$$h(x+1) = \sum_{n=-\infty}^{\infty} f(x+1+n) = \sum_{m=-\infty}^{\infty} f(x+m) = h(x), \quad [1]$$

where we relabelled the dummy index $m = 1 + n$ in the second equality.

(b) Now we calculate Fourier coefficients:

$$\begin{aligned}
c_n &= \frac{1}{T}\int_0^T h(x)e^{-i\omega nx}\,dx && \text{[1] for definition} \\
&= \sum_{m=-\infty}^{\infty}\int_0^1 f(x+m)e^{-2\pi inx}\,dx && \text{since } T = 1,\ \omega = 2\pi \\
&= \sum_{m=-\infty}^{\infty}\int_0^1 f(x+m)e^{-2\pi in(x+m)}\,dx && \text{since } e^{-2\pi inm} = 1 \\
&= \sum_{m=-\infty}^{\infty}\int_m^{m+1} f(x)e^{-2\pi inx}\,dx && \text{[1] for general argument} \\
&= \int_{-\infty}^{\infty} f(x)e^{-2\pi inx}\,dx = F(n). && \text{[1] for definition}
\end{aligned}$$

To prove the Poisson summation formula, simply identify $h(x)$ with its Fourier series:

$$\sum_{n=-\infty}^{\infty} f(x+n) \equiv h(x) = \sum_{n=\infty}^{\infty} c_n e^{2\pi inx} = \sum_{n=\infty}^{\infty} F(n)e^{2\pi inx}. \quad [1]$$

(c) We first calculate the partials of T:

$$\begin{aligned}
\frac{\partial T}{\partial x} &= -\frac{x}{2Dt}T \\
\frac{\partial^2 T}{\partial x^2} &= \left(\frac{x^2}{4D^2t^2} - \frac{1}{2Dt}\right)T \\
\frac{\partial T}{\partial t} &= \left(\frac{x^2}{4Dt^2} - \frac{1}{2t}\right)T. \quad [1]
\end{aligned}$$

Hence,

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2}. \quad [1]$$

Let $a \equiv 4\pi^2 Dt$. Then

$$T(x,t) = \sqrt{\frac{\pi}{a}} e^{-\pi^2 x^2/a} \equiv \delta_a(x),$$

employing equation 8.6 of lectures. As $t \to 0^+$, then $a \to 0^+$ and $T(x,t) \to \delta(x)$ as required. [2]

(d) By linearity, the function $S(\theta,t)$ satisfies the heat equation, since each summand is just a shifted version of T. [1]

For the initial condition, the argument in (c) shows that S converges to a Dirac comb $Ш_T$ of period $T = 1$, i.e. the periodic extension of the Dirac delta. This is equivalent to the Dirac delta $\delta(\theta)$ on a circle of unit circumference. [1]

(e) Since $S(\theta,t) = \sum_n T(\theta + n, t)$, Poisson summation implies

$$S(\theta,t) = \sum_{n=\infty}^{\infty} \mathcal{T}_t(n) e^{2\pi i n\theta} \quad [1]$$

where $\mathcal{T}_t \equiv \hat{\mathcal{F}}[T]$ represents the Fourier transform of T with respect to x, but keeping t fixed. Since T is a Gaussian in the first argument, its Fourier transform is another Gaussian:

$$\mathcal{T}_t(u) = \frac{1}{\sqrt{4\pi Dt}} \cdot \sqrt{4\pi Dt} e^{-4\pi^2 u^2 Dt} = e^{-(2\pi u)^2 Dt}. \quad [2]$$

Hence,

$$S(\theta,t) = \sum_{n=\infty}^{\infty} e^{-(2\pi n)^2 Dt} e^{2\pi i n\theta}.$$

(f) The Greens' functions method expresses $g(\theta)$ as a convolution with the delta function:

$$g(\theta) = \int_0^1 g(\xi)\delta(\theta - \xi)\, d\xi. \quad [1]$$

Since $S(\theta - \xi, t)$ is a fundamental solution to the diffusion equation on a circle (telling us how $\delta(\xi)$ evolves), by linearity we have:

$$\begin{aligned} S_g(\theta,t) &= \int_0^1 g(\xi)\delta(\theta - \xi)\, d\xi \\ &= \sum_{n=-\infty}^{\infty} e^{-(2\pi n)^2 Dt} e^{2\pi i n\theta} \int_0^1 g(\xi) e^{-2\pi i n\xi}\, d\xi \\ &= \sum_{n=-\infty}^{\infty} e^{-(2\pi n)^2 Dt} e^{2\pi i n\theta} d_n. \quad [1] \end{aligned}$$

We used (11) on the second line and the formula for Fourier coefficients on the third.

(g) We observe that

$$K(\theta_1, \theta_2, t) = S(\theta_1, t)S(\theta_2, t) \equiv S_1 S_2, \quad [\mathbf{1}]$$

omitting the functional dependence for convenience. As $t \to 0$, we therefore have

$$K(\theta_1, \theta_2, t) = S_1(t)S_2(t) \to \delta(\theta_1)\delta(\theta_2). \quad [\mathbf{1}]$$

From (d), we know that $\dot{S}_i = D(\partial^2 S_i/\partial\theta_i^2) \equiv DS_i''$, so

$$\dot{K} = \dot{S}_1 S_2 + S_1 \dot{S}_2 = D\big[S_1'' S_2 + S_1 S_2''\big] = D\left(\frac{\partial^2 K}{\partial\theta_1^2} + \frac{\partial^2 K}{\partial\theta_2^2}\right). \quad [\mathbf{1}]$$

Thus, K is the heat kernel on the donut.

(h) **Bonus.** The kernel is now

$$K_N = S_1 \cdots S_N. \quad [\mathbf{1}]$$

The proof that K_N satisfies the PDE [0.5] and the initial condition [0.5] generalises easily (for instance, by induction) from the solution to (g).