Inflation: Inhomogeneities and Spectra

Advanced Seminar, notes by David Wakeham
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Apologies for the length of these notes. To paraphrase Pascal, if I had more time I would have made them shorter. Equations in parentheses (brackets) refer to these notes (Dodelson).

Introduction

Isaac has talked about how inflation solves the problems of large-scale cosmology. Everything in sight was homogeneous. To explain where small-scale, late-time structure comes from, we need to consider inhomogeneous corrections to the inflaton field and the metric. In fact, if we quantise these inhomogeneous corrections, we can explain why the universe has stars, galaxies, and other interesting departures from uniformity! Historically, this was a remarkable and unexpected byproduct. In a slogan: *quantum wrinkles = late-time structure*.

Here's a sneak preview of how it all works. First, figure out how the metric is perturbed during inflation and how the Fourier modes of the perturbations classically evolve. Now quantise those modes and calculate the associated operator uncertainty; this is a cute extension of QFT to de Sitter space. Finally, determine how the modes "classicalise" or decohere as they cross the horizon. This fixes the Fourier spectra of the perturbations. The scalar spectrum sets initial conditions for the Einstein-Boltzmann equation, while the tensor spectrum may leave a trace in the CMB. For particle physics, the take-home message is that we can get surprisingly rich information about physics at inflationary scales from late-time structure, e.g. the CMB. If you want phenomenological constraints for your crazy pet GUT, you could do worse than go outside with a microwave camera and look at the night sky.

Tensor perturbations

Let's recall a few facts about metric perturbations from last week. By the *decomposition theorem*, we have three different types of perturbations which evolve separately (to first order). We expect perturbations of all sorts could be generated by crazy, unknown Planckian physics before inflation. In fact, it's surprising we can't treat these as a perturbation on an FRW background; this is a fine-tuning problem for inflation itself. But that's another story. Vector perturbations quickly decay away in an expanding universe, so we can ignore those and focus on scalar and tensor perturbations. Tensor perturbations are just wobbles in the metric, while scalar perturbations can come both from metric and matter perturbations.

Let's start with tensor perturbations, otherwise known as gravitational waves. In transverse, traceless gauge, the two physical degrees of freedom are manifest: the polarisations h_+ and $h_\times$. You can think of these as the transverse polarisations of a massless spin-2 particle, the graviton. The polarisations $h \in \{h_+, h_\times\}$ obey a wave equation [5.63] in Fourier space

$$\Box h = h'' + 2\mathcal{H}h' + k^2 h = 0, \qquad (1)$$

where $\mathcal{H} \equiv a'/a$ is the *conformal* Hubble parameter and primes (overdots) denote conformal (ordinary) time derivatives.

To massage the wave equation into a nicer form, we switch to the *comoving* tensor perturbation $\tilde{h} \equiv ah$. Since the h' term represents Hubble friction due to the expansion of space, we would expect that this term will vanish for something comoving with the Hubble flow. Dodelson also has a constant we'll ignore for the time being. After a little algebra, you obtain

$$h' = a^{-1}\tilde{h}' - \mathcal{H}\tilde{h}, \quad h'' = a^{-1}\tilde{h}'' - 2a^{-1}\mathcal{H}\tilde{h}' - a^{-2}a''\tilde{h} + 2a^{-1}\mathcal{H}^2\tilde{h}.$$

Substituting these into the wave equation and turning the handle,

$$\begin{aligned}
0 &= h'' + 2\mathcal{H}h' + k^2 h \\
&= a^{-1}\tilde{h}'' - 2a^{-1}\mathcal{H}\tilde{h}' - a^{-2}a''\tilde{h} + 2a^{-1}\mathcal{H}^2\tilde{h} + 2a^{-1}\mathcal{H}\tilde{h}' - 2\mathcal{H}^2\tilde{h} + k^2 a^{-1}\tilde{h} \\
&= a^{-1}\left(\tilde{h}'' - \frac{a''}{a}\tilde{h} + k^2\tilde{h}\right).
\end{aligned}$$

Thus, we have the *Mukhanov-Sasaki equation*

$$\tilde{h}'' + \left(k^2 - \frac{a''}{a}\right)\tilde{h} = 0.$$

As expected, there is no $\tilde{h}'$ term. Squint and you might see a harmonic oscillator with time-dependent frequency $\omega_k^2(\eta) \equiv k^2 - a''/a$.

Scalar perturbations

Now for scalar perturbations. During inflation, scalar perturbations arise from inhomogenous corrections to the inflaton field,

$$\phi(\eta, \mathbf{x}) \equiv \bar{\phi}(\eta) + \delta\phi(\eta, \mathbf{x})$$

where $\bar{\phi}(\eta)$ is the uniform background field. There are also scalar perturbations to the metric, but we can ignore these by choosing *spatially flat gauge* with $\Psi = \Phi = 0$. I will justify this later. For the time being, be thankful and compute.

The basic strategy is to use conservation of stress-energy,

$$\nabla_\mu T^\mu{}_\nu = \partial_\mu T^\mu{}_\nu + \Gamma^\mu_{\alpha\mu} T^\alpha{}_\nu - \Gamma^\alpha_{\nu\mu} T^\mu{}_\alpha = 0. \tag{2}$$

We can expand the stress-energy tensor as $\bar{T} + \delta T$, set $\nu = 0$ and require the first-order part to separately vanish. The Christoffel symbols for perturbed FRW are zero order in spatially flat gauge, with nonvanishing components

$$\Gamma^0_{ij} = g_{ij}H, \quad \Gamma^i_{0j} = \delta^i_j H.$$

Thus, the $\nu = 0$ component of (2) (in Fourier space) becomes

$$\begin{aligned}
0 &= \delta\dot{T}^0{}_0 + ik_i\delta T^i{}_0 + \Gamma^\mu_{\alpha\mu}\delta T^\alpha{}_0 - \Gamma^\alpha_{0\mu}\delta T^\mu{}_\alpha \\
&= \delta\dot{T}^0{}_0 + ik_i\delta T^i{}_0 + 3H\delta T^0{}_0 - H\delta T^i{}_i.
\end{aligned} \tag{3}$$

As we know, the inflaton is a perfect fluid with stress-energy [6.24]:

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu}\left[\frac{1}{2}(\partial\phi)^2 + V(\phi)\right].$$

To first order, we have

$$-(\partial_0\phi)^2 = (\partial\phi)^2 \equiv g^{\mu\nu}\partial_\mu(\bar{\phi}+\delta\phi)\partial_\nu(\bar{\phi}+\delta\phi) = -\dot{\bar{\phi}}^2 - 2\dot{\bar{\phi}}\delta\dot{\phi}.$$

Hence, we get perturbations to the stress-energy tensor

$$\begin{aligned}
T^i{}_0 = g^{i\mu}\partial_\mu(\bar{\phi}+\delta\phi)\partial_0(\bar{\phi}+\delta\phi) \quad &\Longrightarrow \quad \delta T^i{}_0 = a^{-2} i k_i \dot{\bar{\phi}}(\delta\phi) = a^{-3} i k_i \bar{\phi}'(\delta\phi) \\
T^0{}_0 = -(\partial_0\phi)^2 - \frac{1}{2}(\partial\phi)^2 - V(\bar{\phi}+\delta\phi) \quad &\Longrightarrow \quad \delta T^0{}_0 = -\dot{\bar{\phi}}\delta\dot{\phi} - \delta\phi\partial_\phi V = -a^{-2}\bar{\phi}'\delta\phi' - \delta\phi\partial_\phi V \\
T^i{}_j = g^{i\mu}\partial_\mu\phi\partial_j\phi - \delta^i_j\left[\frac{1}{2}(\partial\phi)^2 + V(\bar{\phi}+\delta\phi)\right] \quad &\Longrightarrow \quad \delta T^i{}_j = \delta^i_j\left[a^{-2}\bar{\phi}'\delta\phi' - \delta\phi\partial_\phi V\right].
\end{aligned}$$

We also calculate

$$\delta\dot{T}^0{}_0 = \frac{1}{a}\left(-\frac{1}{a^2}\bar{\phi}'\delta\phi' - \delta\phi\partial_\phi V\right)' = \frac{1}{a^3}\left(2\mathcal{H}\bar{\phi}'\delta\phi' - \bar{\phi}''\delta\phi' - \bar{\phi}'\delta\phi'' - a^2\delta\phi'\partial_\phi V - a^2(\delta\phi)\bar{\phi}'\partial_\phi^2 V\right).$$

Plug everything into (3) and multiply by a^3 to get

$$\begin{aligned}
0 &= 2\mathcal{H}\bar{\phi}'\delta\phi' - \bar{\phi}''\delta\phi' - \bar{\phi}'\delta\phi'' - a^2\delta\phi'\partial_\phi V - a^2(\delta\phi)\bar{\phi}'\partial_\phi^2 V - k^2\bar{\phi}'(\delta\phi) \\
&\qquad - 3\mathcal{H}\left[\bar{\phi}'\delta\phi' + a^2\delta\phi\partial_\phi V\right] - 3\mathcal{H}\left[\bar{\phi}'\delta\phi' - a^2\delta\phi\partial_\phi V\right] \\
&= -\bar{\phi}'\delta\phi'' - \delta\phi'\left(4\mathcal{H}\bar{\phi}' + \bar{\phi}'' + a^2\partial_\phi V\right) - \delta\phi\left(a^2\partial_\phi^2 V + k^2\bar{\phi}\right).
\end{aligned}$$

We can simplify the coefficient of $\delta\phi'$ using the Klein-Gordon equation [6.33] for the background field:

$$\bar{\phi}'' + 2\mathcal{H}\bar{\phi}' + a^2\partial_\phi V = 0.$$

For slow-roll inflation, $\partial_\phi^2 V$ is the same order as the slow-roll parameters, so we neglect it. Dividing by $-\bar{\phi}'$, we finally obtain

$$\delta\phi'' + 2\mathcal{H}\delta\phi' + k^2\delta\phi = 0.$$

What do you know! This is identical to equation (1) for gravity waves. Thus, the *comoving perturbation* $f \equiv a \cdot \delta\phi$ satisfies the Mukhanov-Sasaki equation

$$f'' + \omega_k^2(\eta) f = 0. \tag{4}$$

Incidentally, a quicker way to get this result is to expand the inflaton action out to second order in f and require the variation to vanish order-by-order. The first order part gives [6.33], and the second order part gives (4). As Dodelson mentions, this is annoying to do for gravity waves, but see Baumann, §6.2, for the scalar case.

QFT in de Sitter space

We now need to quantise the comoving perturbations $y \in \{\tilde{h}_+, \tilde{h}_\times, f\}$, which, as we have shown, obey the Mukhanov-Sasaki equation. As Sidney Coleman once said,

> *The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction.*

We will observe this maxim. In fact, at early times, a perturbation of fixed comoving size is much smaller than the curvature scale, so $k^2 \gg a''/a$ and $\omega_k^2(\eta) \approx k^2$. So we have a real harmonic oscillator! This will furnish us with sensible initial conditions.

Generally, we mimic the recipe for a harmonic oscillator in flat space. Dodelson sweeps a lot under the rug, so I'll provide a bit more detail. You can check that Mukhanov-Sasaki follows from the Lagrangian

$$\mathcal{L}_y = \frac{1}{2}\left[(y')^2 - (\nabla y)^2 + \frac{a''}{a}y^2\right].$$

The canonically conjugate momentum is $\pi = \partial \mathcal{L}_y / \partial y' = y'$, so we promote y and π to Hermitian operators satisfying

$$[\hat{y}(\eta, \mathbf{x}), \hat{\pi}(\eta, \mathbf{y})] = i\delta(\mathbf{x} - \mathbf{y}).$$

In Fourier space, this becomes

$$[\hat{y}_{\mathbf{k}}(\eta), \hat{\pi}_{\mathbf{k}'}(\eta)] = i\delta(\mathbf{k} + \mathbf{k}').$$

(The sign change in the delta is not a typo but follows from Hermiticity.) Finally, as with the harmonic oscillator, we introduce time-independent raising and lowering operators $\hat{a}_{\mathbf{k}}$, $\hat{a}_{\mathbf{k}}^\dagger$ and expand $\hat{y}_{\mathbf{k}}$ as

$$\hat{y}_{\mathbf{k}}(\eta) = y_k(\eta)\hat{a}_{\mathbf{k}} + y_k^*(\eta)\hat{a}_{\mathbf{k}}^\dagger.$$

The coefficient $y_k(\eta)$ is called a *mode function*, and it only depends on $k = |\mathbf{k}|$. The same way that the usual flat space mode function $e^{i\omega t}$ satisfies the classical oscillator equation, $y_k(\eta)$ satisfies the Mukhanov-Sasaki equation. This follows from Legendre transforming $\mathcal{L}_y$ and Heisenberg's equation of motion (exercise). The choice of $y_k(\eta)$ and ladder operators is far from unique. In theory, the "vacuum" is the state which minimises the energy, which in turn fixes the ladder operators, but with a time-dependent frequency this is no longer well defined. Swapping between these non-canonical choices is called a *Bogoliubov transformation*, and they are ultimately responsible for thermal radiation from black holes! If you want to learn more, Mukhanov and Winitzki is a pedagogical treasure.

To fix $y_k(\eta)$, first rescale to ensure that

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger] = \delta(\mathbf{k} + \mathbf{k}').$$

To do this, we just divide $y_k(\eta)$ by the *Wronskian* $W[y_k, y_k^*] \equiv -i(y_k y_k^{*\prime} - y_k' y_k^*)$ (exercise). Recall that as $\eta \to -\infty$, all modes satisfy a harmonic oscillator equation. In this case, a unique vacuum state (the *Minkowski vacuum*) minimises the energy, corresponding to the correctly scaled mode function [6.41], or

$$y_k(\eta) \sim \frac{1}{\sqrt{2k}} e^{-ik\eta}, \qquad \eta \to -\infty. \tag{5}$$

The prefactor comes from the harmonic oscillator Wronksian (exercise), while $e^{-ik\eta}$ corresponds to the annihilation operator. This initial condition fixes the time-dependent mode function $y_k(\eta)$. It's called the *Bunch-Davies solution*, and the corresponding vacuum the *Bunch-Davies vacuum*.

Finally, we solve the Mukhanov-Sasaki equation to see how the mode functions evolve. Since H varies slowly, we can make the approximation [6.34] $\eta \approx (aH)^{-1}$, so $a' = a^2 H \simeq -a/\eta$ and

$$\frac{a''}{a} = \frac{(a^2H)'}{a} \simeq -\frac{1}{a}\left(\frac{a}{\eta}\right)' = \frac{a - \eta a'}{a\eta^2} \simeq \frac{2}{\eta^2}.$$

This result is exact in de Sitter space since H is constant. For $\omega_k^2(\eta) = k^2 - 2/\eta^2$ and (5), we can solve the Mukhanov-Sasaki equation to obtain

$$y_k(\eta) = \frac{e^{-ik\eta}}{\sqrt{2k}} \left(1 - \frac{i}{k\eta}\right).$$

Rather than derive this result (it's tedious and involves Bessel functions), let's just check that it works. Clearly, for fixed k and $|k\eta| \gg 1$, we satisfy $y_k(\eta) \sim e^{-ik\eta}/\sqrt{2k}$. It remains to check the equation of motion:

$$y_k'' + \left(k^2 - \frac{2}{\eta^2}\right) y_k = \frac{e^{-ik\eta}}{\sqrt{2k}} \left[\left(-k^2 + \frac{ik}{\eta} + \frac{2}{\eta^2} - \frac{2i}{k\eta^3}\right) + \left(k^2 - \frac{2}{\eta^2}\right)\left(1 - \frac{i}{k\eta}\right)\right] = 0.$$

The de Sitter solution is enough for our purposes.

So, we have the mode expansion

$$\hat{y}(\eta, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}} \big[y_k(\eta)\hat{a}_{\mathbf{k}} + y_k^*(\eta)\hat{a}_{\mathbf{k}}^\dagger\big].$$

The expectation $\langle \hat{y} \rangle = 0$ since the ladder operators kill the vacuum on either side. However, the quantum uncertainty (evaluated at $\mathbf{x} = \mathbf{0}$ for convenience, but anywhere will do) is

$$\begin{aligned}
\langle |\hat{y}|^2 \rangle &= \int \frac{d^3k d^3k'}{(2\pi)^3} \langle 0| (y_k(\eta)\hat{a}_{\mathbf{k}} + y_k^*(\eta)\hat{a}_{\mathbf{k}}^\dagger)(y_{k'}(\eta)\hat{a}_{\mathbf{k}'} + y_{k'}^*(\eta)\hat{a}_{\mathbf{k}'}^\dagger)\big] |0\rangle \\
&= \int \frac{d^3k d^3k'}{(2\pi)^3} y_k(\eta) y_{k'}^*(\eta) \langle 0| [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger] |0\rangle \\
&= \int \frac{d^3k}{(2\pi)^3} |y_k(\eta)|^2 = \int d\ln k \, \frac{k^3}{2\pi^2} |y_k(\eta)|^2 \equiv \int d\ln k \, \Delta_y^2(k, \eta),
\end{aligned}$$

where we have defined the *power spectrum* $\Delta_y^2(k,\eta) \equiv (k^3/2\pi^2)|y_k(\eta)|^2$. I am using the "early universe" spectrum Δ_y^2 in contrast to Dodelson, who uses the "large structure" spectrum $P_y(k,\eta) \equiv (2\pi^2/k^3)\Delta_y^2(k,\eta)$.

You might be wondering how modes decohere or *classicalise* as they leave the horizon. This is a subtle issue, but here is a simple way of looking at it. In the superhorizon limit $k\eta \to 0$, the commutation relations between $\hat{y}$ and its conjugate momentum $\hat{\pi} = \hat{y}'$ break down. More precisely, $y_k' \approx -y_k/\eta$ and hence $\hat{\pi} \propto \hat{y}$. I leave the details to the interested reader.

Primordial gravity waves

Technically, to have canonical kinetic terms for $\tilde{h}$, we actually need to add a factor $(16\pi G)^{-1/2}$: with $\tilde{h}$ defined as above, the coefficient of the kinetic term is $(8\pi G)^{-1}$ from the Einstein-Hilbert action, but we want a coefficient $1/2$. Of course, $\tilde{h}$ still satisfies the Mukhanov-Sasaki equation and the above reasoning is unchanged.

Adding this factor, transforming back to physical perturbations h, and summing over polarisations, we find the power spectrum is

$$\Delta_h^2(k,\eta) \equiv \Delta_{h_+}^2(k,\eta) + \Delta_{h_\times}^2(k,\eta) = \frac{2\cdot 16\pi G}{a^2}\Delta_{\tilde{h}}^2(k,\eta) = \frac{16Gk^3}{\pi a^2}|h_k(\eta)|^2.$$

To determine the spectrum at late times, we evaluate H at horizon crossing when the mode gets frozen. Since $\eta \simeq -1/aH$, the mode crosses the horizon at $k\eta \sim -1$. Using our explicit solution for the mode functions, for $k\eta \lesssim -1$,

$$\Delta_h^2(k,\eta) \sim \frac{16Gk^3}{\pi a^2}\frac{1}{2k^3\eta^2}\bigg|_{k=aH} = \frac{8GH^2}{\pi}\bigg|_{k=aH}.$$

Later, the Hubble radius decreases and these modes reenter the horizon where they continue to evolve. Thus, inflation predicts a background of *primordial gravitational waves*.

If we could measure the gravitational wave spectrum, we would find out about the inflationary Hubble rate, and via the Friedmann equations, the energy scale of inflation. However, the factor G floating around means that unless inflation happens at sufficiently high (Planckian) energies, we are unlikely to directly detect gravitational waves. There are heuristic arguments that suggest the inflationary energy scale is $\sim 10^{15}$ GeV.

One final comment. We expect the CMB to become polarised by interactions with photons (Thomson scattering) before the photons decouple. The polarisation of the photons can be split into E modes (irrotational) and B modes (divergenceless). Scalar perturbations create E modes, while tensor perturbations create both modes, so a detection of B-modes in the CMB is a famous "smoking gun" for inflation. In 2014, the BICEP2 telescope prematurely claimed to have detected these modes, but the signal turned out to be cosmic dust.

Gauge invariance and scalar perturbations

The slickest way to derive the scalar power spectrum is to use gauge-invariant objects. Also, as particle physicists, the term "gauge-invariant" should give you a warm, fuzzy feeling. (Maybe that's just me.) Let's briefly discuss scalar perturbations as per §5.5 of Dodelson. Scalar perturbations have 4 degrees of freedom, A, B, E, ψ (see [5.67]) given by

$$\begin{aligned} g_{00} &= -(1+2A) \\ g_{0i} &= -a\partial_i B \\ g_{ij} &= a^2\bigg(\delta_{ij}[1+2\psi] - 2\partial_i\partial_j E\bigg). \end{aligned}$$

We also have the freedom to change coordinates, via [5.68]

$$t \to \tilde{t} = t + \xi^0(x), \quad x^i \to \tilde{x}^i = x^i + \delta^{ij}\partial_j\xi(x).$$

(You might wonder why we have $\partial_j \xi$ rather than arbitrary ξ^j; it turns that ξ^j must be irrotational to give a scalar perturbation, so we can write it as the grad of a scalar ξ.)

You can show that under these coordinate transformations, the metric perturbations transform as

$$\begin{aligned} A &\to \tilde{A} = A - \frac{1}{a}\dot{\xi}^0 \\ \psi &\to \tilde{\psi} = \psi - H\xi^0 \\ B &\to \tilde{B} = B - \frac{1}{a}\xi^0 + \dot{\xi} \\ E &\to \tilde{E} = E + \xi. \end{aligned}$$

With four metric perturbations (A, B, E, ψ) and two degrees of freedom in coordinate transformations (ξ^0, ξ), we only have two genuine degrees of freedom. Bardeen wrote them down in a gauge-invariant way:

$$\Phi_A \equiv A + \frac{1}{a}[a(\dot{E} - B)]', \qquad \Phi_H \equiv -\psi + aH(B - \dot{E}).$$

For instance, we can check that Φ_H is gauge-invariant:

$$\Phi_H \to \tilde{\Phi}_H = -\tilde{\psi} + aH(\tilde{B} - \dot{\tilde{E}}) = -\psi + H\xi^0 + aH(B - a^{-1}\xi^0 + \dot{\xi} - \dot{E} - \dot{\xi}) = \Phi_H.$$

Bardeen also constructed two gauge-invariant matter perturbations:

$$v \equiv ikB + \frac{\hat{k}^i \delta T^0{}_i}{(\rho + P)a}, \qquad \epsilon_m \equiv -1 - \frac{1}{\rho}\delta T^0{}_0 + \frac{3H}{k^2\rho}k^i \delta T^0{}_i.$$

You can think of these as "velocity" and energy perturbations. I won't prove these are gauge-invariant; again, see §4.2 of Baumann for details.

Now, to get spatially flat gauge just change coordinates so that $E = \psi = 0$ and the spatial part of the metric $g_{ij} = a^2\delta_{ij}$ is unperturbed. As claimed earlier, this clearly implies $\Psi = \Phi = 0$. We would like to figure out a linear combination of Bardeen variables which is proportional to $\delta\phi$. Using our earlier result for $\delta T^0{}_i$, we find that

$$v = ikB - \frac{ik\bar{\phi}'\,\delta\phi}{(\rho + P)a^2}, \qquad \Phi_H = aHB.$$

Thus, a linear combination that works is the *comoving curvature perturbation*

$$\zeta \equiv -\Phi_H - \frac{iaH}{k}v = -\frac{aH}{\bar{\phi}'}\delta\phi.$$

Physically, Φ_H sources curvature inhomogeneities in spatial 3-slices, or $4a^{-2}k^2\Phi_H = {}^{(3)}\mathcal{R}$. In *comoving gauge* where $v = 0$, $\zeta = \Phi_H$, hence the name I've given ζ.

The spectrum of $\delta\phi$ is exactly the same as h, except for the factor of $2 \cdot 16\pi G$, so

$$\Delta^2_{\delta\phi}(k, \eta) = \left.\frac{H^2}{8\pi^2}\right|_{k=aH}.$$

To find the spectrum of ζ, recall the slow-parameter $\epsilon \equiv -\dot{H}/H^2$. We note that during slow-roll, the second Friedmann equation implies $\dot{H} = -4\pi G\dot{\phi}^2 = -4\pi G(\phi'/a)^2$, and hence

$$\left(\frac{aH}{\dot{\bar{\phi}}}\right)^2 = \frac{4\pi G a^2 H^2}{4\pi G(\bar{\phi}')^2} = \frac{4\pi G}{\epsilon}.$$

Thus, we obtain

$$\Delta_\zeta^2(k,\eta) = \left(\frac{aH}{\bar{\phi}^2}\right)^2 \frac{H^2}{8\pi^2}\bigg|_{k=aH} = \frac{GH^2}{2\pi\epsilon}\bigg|_{k=aH}$$

On superhorizon scales, ζ is conserved. Later the modes reenter the horizon and set initial conditions for the Einstein-Boltzmann equation.

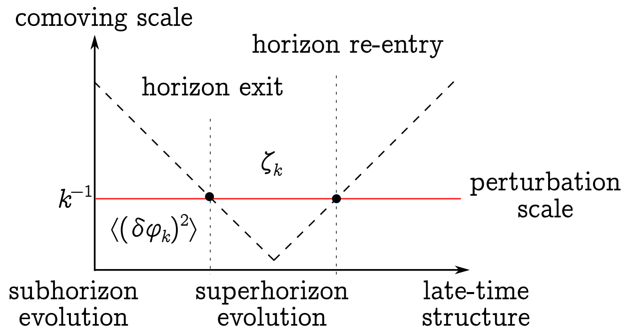

Figure 1: Inflation and comoving scale.

Conservation of ζ on superhorizon scales is not trivial, and uninsightful proofs may be found in Dodelson, §6.5.2 and Baumann, §4.3. However, the intuition behind the result is straightforward: the mode cannot evolve since it is too large to interact with itself. This way of thinking is formalised by the "separate universes" proof due to Wands et al. (arXiv:astro-ph/0003278).
Proof sketch. Take two disjoint superhorizon-sized patches. In spatially flat gauge, these patches are locally uniform, with ζ simply deforming the time values to ensure the curvature across the patch cancels out. This is creates a position-dependent offset to the cosmological time on the the patch. The patches evolve independently like separate FRW universes, unable to interact since they are superhorizon, with the ζ-deformed offsets propagating forward without change. So ζ is conserved! ■

Just to summarise all the variable juggling, we started with the inflaton perturbation $\delta\phi$; we moved to the comoving perturbation $f = a \cdot \delta\phi$ since it was easier to quantise; we ended with $\zeta = -(aH/\bar{\phi}')\delta\phi$ since it is gauge-invariant and conserved on superhorizon scales.

Spectral indices and summary

A spectrum Δ_y^2 is *scale-invariant* or *Gaussian* if it is independent of k. This would be the case if H was constant. Since H varies, near a reference scale or *pivot* k_*, we anticipate a power-law dependence which is almost scale-invariant:

$$\Delta_\zeta^2(k) \equiv A_s \left(\frac{k}{k_*}\right)^{n_s-1}.$$

I'll sketch a proof below. Here, $n_s \equiv 1 + d\ln\Delta_\zeta^2/d\ln k$ is the *scalar spectral index*, quantifying the departure from scale-invariance. For $k_* = 0.05$ Mpc^{-1}, measurements show that

$$A_s = (2.196 \pm 0.060) \times 10^{-9}, \quad n_s = 0.9604 \pm 0.0073.$$

Of course, only A_s depends on the pivot k_*. We get something similar for tensor perturbations:

$$\Delta_h^2(k) \equiv A_t \left(\frac{k}{k_*}\right)^{n_t}.$$

Note that the tensor spectral index n_t is defined without the 1. The ratio of spectral amplitudes is called the *tensor-to-scalar ratio*. It is approximately the ratio of amplitudes, and an upper bound (from non-observation of a primordial background) is

$$r \equiv \frac{A_t}{A_s} \simeq \frac{\Delta_h^2}{\Delta_\zeta^2} \simeq 16\epsilon, \quad r \lesssim 0.11.$$

Let's see why power law behaviour is expected and how the spectral indices encode inflationary parameters. First, observe that

$$\frac{d}{dt}(aH)^{-1} = -\left(\frac{1}{a} + \frac{\dot{H}}{aH^2}\right) \equiv -\frac{1}{a}(1-\epsilon) < 0$$

where ϵ is the fractional change in H per inflationary e-fold defined above. More conveniently, it is

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = -\frac{d\ln H}{dN},$$

where $N \equiv \ln a$ counts e-folds. Another convenient slow-roll parameter is ς, the fractional change in ϵ per e-fold:

$$\varsigma \equiv \frac{d\ln\epsilon}{dN}.$$

(This is usually called η, but that Greek letter is spoken for.) Using the fact that $\ln k = N + \ln H$ at horizon crossing, to first-order in ϵ, ς we have

$$\begin{aligned}
n_s - 1 &= \frac{d\ln\Delta_\zeta^2}{dN}\frac{dN}{d\ln k} \\
&= \left(2\frac{d\ln H}{dN} - \frac{d\ln\epsilon}{dN}\right)\left(1 + \frac{d\ln H}{dN}\right)^{-1} \\
&= (-2\epsilon - \varsigma)(1+\epsilon) + \mathcal{O}(\epsilon^2) \\
&= -2\epsilon - \varsigma + \mathcal{O}(\epsilon^2).
\end{aligned}$$

The same argument (omitting ς) shows that $n_t \simeq -2\epsilon$. So power-law behaviour results for small, constant Hubble slow-roll parameters. To recover Dodelson's result, we use the fact that

$$\varsigma = \frac{\dot{\epsilon}}{H\epsilon} = -4\left(\frac{\dot{H}}{H^2}\right) - 2\left(\frac{H\ddot{\phi} + \dot{H}\dot{\phi}}{H^2\dot{\phi}}\right) \simeq 4\epsilon - 2(\epsilon - \delta) = 2(\epsilon + \delta).$$

It follows that

$$n_s - 1 \simeq -4\epsilon - 2\delta.$$

To summarise, the spectrum of scalar perturbations provides initial conditions for late-time structure, while tensor perturbations could exist as a primordial gravitational wave background (unlikely) or the polarisation of the CMB (smoking gun for inflation). Running things in reverse, late-time structure (particularly the spectral indices) places phenomenological constraints on inflation. Since inflation is expected to be a GUT-scale phenomenon ($\sim 10^{15}$ GeV), early universe cosmology could have lots to tell us about high energy physics!

References

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