Maxwell's Demon goes to Vegas

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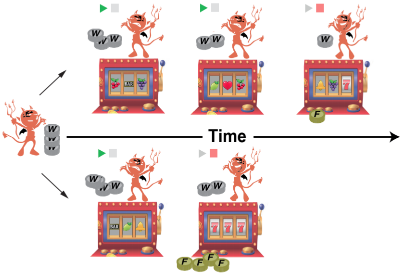
Thermodynamic setup

- Maxwell's demon goes to Vegas to play slot machines. Playing costs work W but yields free energy ΔF.



- Demon's goal: beat second law in form $\Delta F \leq W$.
- Note: second law in Clausius form ($\Delta S \geq Q/T$) gives

$$W - \Delta F = W - \Delta U + T\Delta S = -Q + T\Delta S \geq 0.$$

Lindblad equation

- A slot machine is a system coupled to an environment.
- The system density ρ obeys the Lindblad master equation:

$$\dot{\rho} = -i[H, \rho] + \sum_k L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\}.$$

- Let's explain the terms:
  - The first term is evolution within the system.
  - The L_k, called jump operators, are Kraus operators for infinitesimal time evolution.
  - The last term ensures preservation of trace.
- For now, take the dynamics (H, L_k) to be time-invariant.

Good detectors

- The Lindblad equation deterministically evolves ρ like Fokker-Planck for a classical dissipative system. Is there an analog of Langevin for individual random trajectories?
- The $\{L_k\}$ are POVM for environment to measure system.
- We assume the environment is a good detector, and tells us when (t_j) and what (k_j) measurements it makes.

Stochastic Schrödinger equation (SSE)

- If the system is in a pure state $|\psi\rangle$, it can either (a) evolve smoothly, or (b) get measured, $|\psi\rangle \to L_k|\psi\rangle$.
- For (a), we use a Hamiltonian with couplings removed:

$$H' = H - \frac{i}{2}\sum_k L_k L_k^\dagger.$$

- Combining gives the stochastic Schrödinger equation:

$$d|\tilde{\psi}(t)\rangle = -iH'\,dt|\psi(t)\rangle + \sum_k dN_k(L_k - 1)|\psi(t)\rangle,$$

where dN_k is Poisson, and $|\tilde{\psi}(t)\rangle$ is unnormalized.

A (demon-free) slot machine

- The casino announces an initial density $\rho(0)$, which evolves by the Lindblad equation. We write

$$\rho(t) = \sum_n p_n(t)|n(t)\rangle\langle n(t)|.$$

- The casino initializes $|\psi(0)\rangle = |n(0)\rangle$ with probability $p_n(0)$, and evolves via SSE with jumps $\mathcal{R} = \{(k_j, t_j)\}$.
- Without demon, at τ machine projects onto $\rho(\tau)$ basis:

$$\rho(0) \overset{p_n(0)}{\longrightarrow} |n(0)\rangle \overset{\mathbb{P}[\mathcal{R}_0^\tau|n(0)]}{\longrightarrow} |\psi(\tau)\rangle \overset{|\langle\psi(\tau)|m(\tau)\rangle|^2}{\longrightarrow} |m(\tau)\rangle.$$

Stopping conditions

- Let's add a demon. They apply a stopping condition to the jump record $\mathcal{R}_0^t$ to determine if the game stops.
- If they terminate at $t \leq \tau$, machine projects onto $\rho(t)$:

$$\rho(0) \overset{p_n(0)}{\longrightarrow} |n(0)\rangle \overset{\mathbb{P}[\mathcal{R}_0^t|n(0)]}{\longrightarrow} |\psi(t)\rangle \overset{|\langle\psi(t)|m(t)\rangle|^2}{\longrightarrow} |m(t)\rangle.$$

- A trajectory is the jump record plus projective bookends:

$$\gamma_{\{0,t\}} = (n, \mathcal{R}_0^t, m).$$

Trajectory probabilities

- The stochastic wavefunction $|\psi(t)\rangle$ is generated by a record-dependent operator $\mathcal{L}_{\mathcal{R}}$:

$$|\psi(t)\rangle = \frac{\mathcal{L}_{\mathcal{R}_0^t}|n(0)\rangle}{\sqrt{\langle \mathcal{L}^\dagger_{\mathcal{R}_0^t}\mathcal{L}_{\mathcal{R}_0^t}\rangle_{n(0)}}}.$$

  The probability of the record appearing is $\langle \mathcal{L}^\dagger_{\mathcal{R}_0^t}\mathcal{L}_{\mathcal{R}_0^t}\rangle_{n(0)}$.

- The probability of $\gamma_{\{0,t\}} = (n, \mathcal{R}_0^t, m)$ is then

$$\mathbb{P}[\gamma_{\{0,t\}}] = p_n(0)|\langle m(t)|\mathcal{L}_{\mathcal{R}_0^t}|n(0)\rangle|^2.$$

- Similarly, for the reversed trajectory $\tilde{\gamma}_{\{0,t\}} = (m, \tilde{\mathcal{R}}_t^0, n)$,

$$\mathbb{P}[\tilde{\gamma}_{\{0,t\}}] = p_m(t)|\langle n(0)|\Theta^\dagger \mathcal{L}_{\tilde{\mathcal{R}}_t^0}\Theta|m(t)\rangle|^2.$$

Detailed balance

- The record operator $\mathcal{L}_{\mathcal{R}}$ has a detailed balance relation:

$$\Theta^{\dagger}\mathcal{L}_{\tilde{\mathcal{R}}}\Theta = e^{-\Delta S_{\text{env}}(\mathcal{R})/2}\mathcal{L}_{\mathcal{R}}^{\dagger},$$

where $\Delta S_{\text{env}}(\mathcal{R})$ is the change in environmental entropy.

- (S_{env} just sums $-\ln p_J^E$ for environmental pointer states.)
- Detailed balanced implies

$$|\langle n(0)|\Theta^{\dagger}\mathcal{L}_{\tilde{\mathcal{R}}_{\mathcal{T}}^{0}}\Theta|m(\mathcal{T})\rangle|^2 \\ = e^{-\Delta S_{\text{env}}(\mathcal{R})}|\langle m(\mathcal{T})|\mathcal{L}_{\mathcal{R}_0^{\mathcal{T}}}|n(0)\rangle|^2.$$

Crooks and Jarzynski

- Let's reconnect to thermodynamics and warm up with the Crooks fluctuation theorem. Our work so far gives

$$\Delta S_{\text{tot}}(t) = \ln \frac{\mathbb{P}[\gamma_{\{0,t\}}]}{\mathbb{P}[\tilde{\gamma}_{\{0,t\}}]} = \ln \frac{p_n(0)}{p_m(t)} + \Delta S_{\text{env}}(\mathcal{R}).$$

- Exponentiating and averaging gives the Jarzynski equality:

$$\langle e^{-\Delta S_{\text{tot}}(t)} \rangle = \sum_{\gamma} \mathbb{P}[\gamma_{\{0,t\}}] \cdot \frac{\mathbb{P}[\tilde{\gamma}_{\{0,t\}}]}{\mathbb{P}[\gamma_{\{0,t\}}]} = 1,$$

since the γ also enumerate the $\tilde{\gamma}$.

The second law at fixed times

- From Jensen's inequality,

$$1 = \langle e^{-\Delta S_{\text{tot}}(t)} \rangle \geq e^{-\langle \Delta S_{\text{tot}}(t) \rangle} \implies \langle \Delta S_{\text{tot}}(t) \rangle \geq 0.$$

  But

$$\langle \ln p_n \rangle = \sum_n p_n \ln p_n = -S_{\text{sys}}(0),$$

  and $-\langle p_m \rangle = S_{\text{sys}}(t)$. So $\langle \Delta S_{\text{tot}} \rangle = \langle \Delta S_{\text{sys}} \rangle + \langle \Delta S_{\text{env}} \rangle$.

- For a thermal reservoir at fixed β, $\Delta S_{\text{env}} = -\beta Q$, and we recover the second law:

$$\langle \Delta S_{\text{tot}} \rangle = \langle \Delta S_{\text{sys}} - \beta Q \rangle = \beta \langle W - \Delta F \rangle \geq 0.$$

A mathemagic trick

- So, the demon cannot beat the second law for fixed $\mathcal{T}$. But what about random $\mathcal{T}$ from a stopping condition?
- Let's re-split the total entropy production ΔS_{tot} as

$$\Delta S_{\text{tot}} = \Delta S_{\text{unc}} + \Delta S_{\text{mar}},$$

where ΔS_{unc} is quantum measurement uncertainty,

$$\Delta S_{\text{unc}}(t) = -\ln \frac{p_m(t)}{\langle \rho(t) \rangle_{\psi(t)}},$$

and $\quad \Delta S_{\text{mar}}(t) = \ln \frac{p_n(0)}{\langle \rho(t) \rangle_{\psi(t)}} + \Delta S_{\text{env}}.$

Martingales for time-invariant dynamics

- ΔS_{mar} is mathemagical rather than physical. For any $t \leq t' \leq \tau$, and time-invariant dynamics, one can prove

$$\langle e^{-\Delta S_{\text{mar}}(t')} | (n, \mathcal{R}_0^t) \rangle = e^{-\Delta S_{\text{mar}}(t)}.$$

  Note that $(n, \mathcal{R}_0^t)$ determines $e^{-\Delta S_{\text{mar}}(s)}$ for any $s \leq t$, but the converse also holds, i.e. we can write

$$\langle e^{-\Delta S_{\text{mar}}(t')} | e^{-\Delta S_{\text{mar}}(s)}, 0 \leq s \leq t \rangle = e^{-\Delta S_{\text{mar}}(t)}.$$

- This defines a martingale, a process whose expectation, conditioned on the past, is the most recent observation:

$$\langle X(t') | X(s), 0 \leq s \leq t \rangle = X(t), \quad \text{for } t' \geq t.$$

Doobious outcomes

- Martingales enjoy many beautiful properties.
- One is Doob's optional stopping theorem, stating that applying a stopping condition can't improve your return:

$$\langle X(\mathcal{T})\rangle_{\mathcal{T}} = \langle X(0)\rangle.$$

- In our case, this immediately gives a fluctuation theorem:

$$\langle e^{-\Delta S_{\mathrm{mar}}(\mathcal{T})}\rangle_{\mathcal{T}} = \langle e^{-\Delta S_{\mathrm{mar}}(0)}\rangle = 1,$$

since $\Delta S_{\mathrm{env}} = 0$ and $|\psi(0)\rangle = |n(0)\rangle$.

The second law for stopping

- As before, Jensen's inequality gives $\langle \Delta S_{\mathrm{mar}}(\mathcal{T}) \rangle_{\mathcal{T}} \geq 0$. Since $\Delta S_{\mathrm{tot}} = \Delta S_{\mathrm{unc}} + \Delta S_{\mathrm{mar}}$, we get a new second law

$$\langle \Delta S_{\mathrm{tot}}(\mathcal{T}) \rangle_{\mathcal{T}} = \beta \langle W - \Delta F \rangle_{\mathcal{T}} \geq \langle \Delta S_{\mathrm{unc}}(\mathcal{T}) \rangle_{\mathcal{T}}.$$

- Can this violate the usual second law? We have

$$\Delta S_{\mathrm{unc}}(t) = -\ln \frac{p_m(t)}{\sum_{m'} p_{m'}(t) |\langle m'(t) | \psi(t) \rangle|^2}.$$

  This can be positive or negative! So a well-chosen stopping condition can violate the second law.

Classical limit

- Unfortunately, this effect is purely quantum.
- In the classical limit, the record $\mathcal{R}$ becomes a classical trajectory $\{(x(t), t)\}$, with $|\psi(t)\rangle = |x(t)\rangle$ always an eigenvector of $\rho(t)$. This means $\Delta S_{\text{unc}} = 0$, and hence

$$\langle \Delta S_{\text{tot}} \rangle_{\mathcal{T}} = \langle \Delta S_{\text{mar}} \rangle_{\mathcal{T}} = \beta \langle W - \Delta F \rangle_{\mathcal{T}} \geq 0.$$

- To get classical violations, we must use nonequilibrium driving, i.e. time-dependent parameters:

$$H(\lambda_t), L_k(\lambda_t) \quad \text{for} \quad 0 \leq t \leq \tau.$$

Driving and asymmetry

- The slot machine has forward process $\rho(t)$. We can run the backward process $\Theta^\dagger \tilde{\rho}(t)\Theta$, with relations

$$\Theta^\dagger \tilde{\rho}(0)\Theta = \rho(\tau), \quad \Theta^\dagger \tilde{\rho}(\tau)\Theta = \rho(0).$$

- We define the asymmetry as the log ratio

$$\delta(t) = \ln \frac{\langle \rho(t) \rangle_{\psi(t)}}{\langle \Theta^\dagger \tilde{\rho}(\tau - t)\Theta \rangle_{\psi(t)}}.$$

- Without driving, this vanishes by microscopic reversibility and invariance of dynamics. But with driving, you can tell them apart. In fact, at fixed time, $\langle \delta(t) \rangle = S(\rho(t) || \tilde{\rho}(t))$.

Martingales for driven systems

- For driven systems, our old martingale must be modified:

$$\langle e^{-\Delta S_{\mathrm{mar}}(t')-\delta(t')} | (n, \mathcal{R}_0^t) \rangle = e^{-\Delta S_{\mathrm{mar}}(t)-\delta(t)}.$$

- Doob gives a fluctuation theorem:

$$\langle e^{-\Delta S_{\mathrm{mar}}(\mathcal{T})-\delta(\mathcal{T})} \rangle_{\mathcal{T}} = \langle e^{-\Delta S_{\mathrm{mar}}(0)-\delta(0)} \rangle = 1.$$

- Jensen's inequality implies a modified second law:

$$\langle \Delta S_{\mathrm{mar}}(\mathcal{T}) \rangle_{\mathcal{T}} \geq -\langle \delta(\mathcal{T}) \rangle_{\mathcal{T}}.$$

A second law for driven stopping

- In the classical limit, $\Delta S_{\text{tot}} = \Delta S_{\text{mar}}$ as before.
- But δ does not vanish! Instead, it becomes the asymmetry for classical trajectories:

$$\delta(t) = \ln \frac{\langle \rho(t) \rangle_{\psi(t)}}{\langle \tilde{\rho}(\tau - t) \rangle_{\Theta\psi(t)}} \to \ln \frac{p[x(t), t]}{\tilde{p}[x(t), \tau - t]}.$$

- Thus, the second law for classical driven stopping is

$$\langle \Delta S_{\text{tot}} \rangle_{\mathcal{T}} = \beta \langle W - \Delta F \rangle_{\mathcal{T}} \geq -\langle \delta \rangle_{\mathcal{T}}.$$

At fixed times, $\langle \delta \rangle = D(p || \tilde{p}) \geq 0$. Even stopping at a fixed time can violate the second law for driven systems!

Comments and questions

- What happens with basic feedback? E.g. a slot machine where the demon can project onto $\rho(t)$ when it likes.
- Ultimately, second law violations come from black boxes in the system. What are the hidden entropy costs of driving, or implementing a stopping strategy?
- Presumably: increase in correlations with the driver (for classical stopping) or demon (for quantum stopping).
- Any questions? Thanks for listening!

References

- "Thermodynamics of Gambling Demons" (2020), Manzano, Subero, Maillet, Fazio, Pekola and Roldan.
- "Quantum Martingale Theory and Entropy Production" (2019), Manzano, Fazio and Roldan.
- "Quantum fluctuation theorems for arbitrary environments" (2017), Manzano, Horowitz and Parrondo.
- "Dissipation: The phase-space perspective" (2007), Kawai, Parrondo and Van den Broeck.