PHYC20014 Physical Systems

Wave Theory and Fourier Analysis: Tutorial 1

Tutorial problems

1. **Basic Fourier series.** We start with some "get to know you" exercises. Sketch the periodic extension of these functions and calculate their Fourier series.

   (a) The *square wave* $\Pi : (-\pi, \pi] \to \mathbb{R}$:

   $$\Pi(\theta) = \begin{cases} +1 & \theta > 0 \\ 0 & \theta = 0 \\ -1 & \theta < 0. \end{cases}$$

   (b) The *triangle wave* $\Delta : (-\pi, \pi] \to \mathbb{R}$:

   $$\Delta(\theta) = \pi - |\theta|.$$

2. **Derivatives.** Check that (where differentiable) $\Delta' = -\Pi$. Differentiating term-by-term, verify this relation also holds for the associated Fourier series.

3. **Sine, cosine and half-range.** An *odd* function has the property that $f(-\theta) = -f(\theta)$, while an *even* function satisfies $f(-\theta) = f(\theta)$.

   (a) Show that for odd (even) functions, the Fourier coefficients a_n (b_n) vanish. Since the cosine terms vanish, an odd function has a *sine series*, and similarly, an even function has a *cosine series*.

   (b) Prove that you can uniquely split an arbitrary function f into odd and even parts:

   $$f(\theta) = f_+(\theta) + f_-(\theta), \quad f_\pm(-\theta) = \pm f_\pm(\theta).$$

   Thus, the Fourier series for f splits into a cosine series for f_+ and a sine series for f_-.

   (c) Recall from lectures that the *half-range expansion* of a function is a Fourier series valid over $[0, L]$. We can use either a cosine series (a_n terms) or sine series (b_n terms). Find both half-range expansions for $f(\theta) = e^\theta - 1$ on $[0, \pi]$, and comment on the difference.

4. **Dirac comb.** The *Dirac delta function* $\delta(t)$ models an infinitely strong point impulse, and is defined by the "sifting" property

   $$\int_{-\infty}^{\infty} \delta(t) f(t)\, dx = f(0).$$

   The *Dirac comb* Ш_T is a periodically repeating version with period T:

   $$\text{Ш}_T(\theta) \equiv \sum_{k=-\infty}^{\infty} \delta(\theta - kT).$$

   (a) For Ш_T, derive the Fourier series representation

   $$\text{Ш}_T(\theta) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{i\omega n\theta}, \quad \omega \equiv \frac{2\pi}{T}.$$

(b) *Check that the series representation of $\text{Ш}_T(\theta)$ blows up at multiples of T. Show that, at other points, we get infinitely fast oscillations.[1]

5. **Numerical series.** Evaluating a Fourier series at a well-chosen point sometimes yields nontrivial mathematical results. Earlier, you should have found that $\Pi(\theta)$ has the Fourier series

$$\Pi(\theta) = \sum_{n=0}^{\infty} \frac{4}{(2n+1)\pi} \sin[(2n+1)\theta].$$

By wisely choosing a point to evaluate both sides, prove Leibniz's formula for π:

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Extra problems

6. **Exponential Fourier series.** For a function with period T, we usually write

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos(\omega n\theta) + b_n \sin(\omega n\theta) \Big], \quad \omega = \frac{2\pi}{T}.$$

Recall from lectures that we can write the same series using complex exponentials:

$$f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega\theta}.$$

Determine the relationship between c_n and the a_n, b_n for a *real* function.

7. **Chebyshev polynomials.** You may recall de Moivre's theorem from high school:

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

Expand the LHS and take the real part of both sides. You will get some linear combination of products of powers of $\cos\theta$ and *even* powers of $\sin\theta$; converting the latter to cosines using $\sin^2\theta = 1 - \cos^2\theta$, the end result is an expression for $\cos(n\theta)$ which is a *polynomial* in $\cos\theta$:

$$\cos(n\theta) \equiv T_n(\cos\theta), \quad n = 0, 1, 2, \dots$$

The polynomials T_n are called *Chebyshev polynomials*. Since $T_n(\cos\theta) = \cos(n\theta)$, Chebyshev polynomials are related to cosine series (Problem 4). Making the change of variable $x = \cos\theta$, use results about Fourier series to show that

$$\int_{-1}^{1} \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}} dx = \begin{cases} \frac{\pi}{2} & m = n \text{ and } m, n \geq 1, \\ \pi & m = n = 0, \\ 0 & \text{else.} \end{cases}$$

[1] These infinite oscillations vanish in the sense of *generalised functions*. Understanding what this means rigorously is beyond the scope of the course.

8. **The Gibbs phenomenon.** NOTE: *This problem requires a computer.* Any finite sum of trigonometric functions is continuous. Thus, any partial sum in the Fourier series for a *discontinuous* function is fundamentally different from the function that it represents. This leads to some rather strange behaviour in the convergence of Fourier series, as we'll now see.

   (a) Recall that the Fourier series for $\Pi(\theta)$ is

   $$\Pi(\theta) = \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi} \sin[(2n-1)\theta].$$

   Define the *partial Fourier series*

   $$\Pi_N(\theta) \equiv \sum_{n=1}^{N} \frac{(-1)^n}{2n-1} \sin[(2n-1)\theta].$$

   Using a computer, plot Π_N for $N = 5, 10, 50, 100$.

   (b) You should observe an "overshoot" in Π_N at the discontinuities of Π. This is called the *Gibbs phenomenon.* Does the size of the overshoot appear to change with N? Estimate how large it is compared to the underlying discontinuity. (You should find that the jump is $\sim 8.95\%$ the size of the discontinuity.)

   (c) We can try to eliminate the Gibbs phenomenon as follows. Define the *σ-approximated* series for a function f as

   $$f_N^k(\theta) \equiv \frac{a_0}{2} + \sum_{n=1}^{N} \operatorname{sinc}\left(\frac{n}{N}\right)^k \Big[a_n \cos(\omega n\theta) + b_n \sin(\omega n\theta)\Big],$$

   where $\omega \equiv 2\pi/T$ as usual, and

   $$\operatorname{sinc}(x) \equiv \frac{\sin(\pi x)}{\pi x}.$$

   Play around with different values of N and k (using your own code or `gibbs.nb`) and see what happens. What is the tradeoff for suppressing the overshoot?

9. **The Basel problem.** The triangle wave $\Delta(\theta)$ from 1(b) has Fourier series

   $$\Delta(\theta) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{4}{(2n-1)^2\pi} \cos\left[(2n-1)\theta\right].$$

   (a) By a judicious selection of θ, show that

   $$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

   (b) Let B denote the sum of reciprocal squares,

   $$B \equiv \sum_{n=0}^{\infty} \frac{1}{n^2}.$$

Determining B is called the *Basel problem*, and was first posed by in 1644. It took almost 100 years and the genius of LEONARD EULER (1707–1783) to solve it. Show that

$$B = \frac{1}{4}B + \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2},$$

and use your result from (a) to conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

10. **Harmonic symmetry.*** We can generalise the decomposition into odd and even parts (Problem 3) as follows. Consider a complex map $f : \mathbb{C} \to \mathbb{C}$, and fix a natural number n. Let $\omega \equiv e^{2\pi i/n}$ be an n-th root of unity, so that $\omega^n = 1$. Define

$$f_j(z) \equiv \frac{1}{n} \sum_{k=0}^{n-1} f(\omega^k z)\omega^{-jk}.$$

(a) Show that $f_j(\omega^m z) = \omega^{jm} f_j(z)$, and that

$$f(z) = \sum_{j=0}^{n-1} f_j(z).$$

HINT. Recall the formula for geometric sums,

$$\sum_{j=0}^{n-1} \omega^{-jk} = \frac{1 - \omega^{-kn}}{1 - \omega^{-k}}, \quad k \neq 0.$$

(b) Check that setting $n = 2$ reproduces the results in Problem 3(b).

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Solutions

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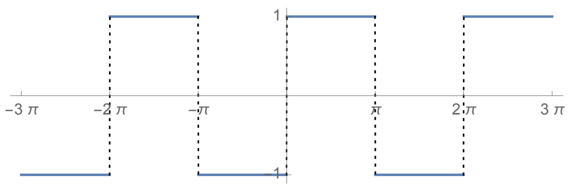
1. **Basic Fourier series.** We use the usual formulas:

$$a_n = \frac{1}{L}\int_{-L}^{L} \cos\left(\frac{\pi n\theta}{L}\right) f(\theta)\, d\theta, \quad b_n = \frac{1}{L}\int_{-L}^{L} \sin\left(\frac{\pi n\theta}{L}\right) f(\theta)\, d\theta,$$

setting $L = \pi$. We also exploit the results of Problem 4.

(a) First, let's graph the periodic extension:



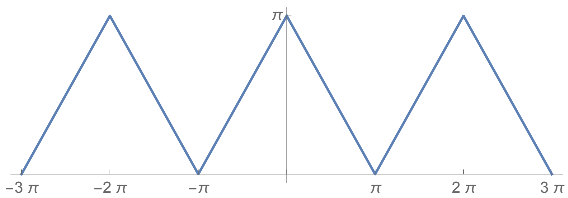
For the square wave, the a_n vanish by symmetry (see Problem 4). The remaining sine terms are given by

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(n\theta)\, \Pi(\theta)\, d\theta \\ &= \frac{2}{\pi}\int_{0}^{\pi} \sin(n\theta)\ d\theta \\ &= -\frac{2}{n\pi}\Big[\cos(n\theta)\Big]_0^{\pi} \\ &= \frac{4}{n\pi}[1-(-1)^n] = \begin{cases} 4/n\pi & n \text{ odd} \\ 0 & n \text{ even.} \end{cases} \end{aligned}$$

Hence, the Fourier series for the periodic extension of Π is

$$\Pi(\theta) = \sum_{n=1}^{\infty} b_n \sin(n\theta) = \sum_{n=0}^{\infty} \frac{4}{(2n+1)\pi}\sin[(2n+1)\theta].$$

(b) We begin with a graph:



For the triangle wave, the b_n vanish by symmetry (see Problem 4). From geometry, the area under the curve $a_0 = \pi$. This leaves

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(n\theta)\,\Delta(\theta)\,d\theta \\
&= \frac{2}{\pi}\int_{0}^{\pi} \cos(n\theta)\,(\pi-\theta)\,d\theta \\
&= \frac{2}{n\pi}\left[\sin(n\theta)(\pi-\theta)\right]_0^{\pi} + \frac{2}{n\pi}\int_0^{\pi}\sin(n\theta)\ d\theta \\
&= -\frac{2}{n^2\pi}\left[\cos(n\theta)\right]_0^{\pi} = \begin{cases} 4/n^2\pi & n \text{ odd} \\ 0 & n \text{ even.}\end{cases}
\end{aligned}$$

Hence, the Fourier series for the periodic extension of Δ is

$$\Delta(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\theta) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{4}{(2n-1)^2\pi}\cos[(2n-1)\theta].$$

2. **Derivatives.** On $(-\pi,\pi]$, $\Delta(\theta) = \pi - |\theta|$, and hence

$$\Delta'(\theta) = \begin{cases} +1 & -\pi < \theta < 0 \\ -1 & 0 < \theta < \pi. \end{cases}$$

Hence, $\Delta' = -\Pi$ on $(-\pi,0)\cup(0,\pi)$. This also clearly holds for the periodic extensions, except at multiples of π where Δ is not differentiable. Now we differentiate the Fourier series for Δ:

$$\frac{d}{d\theta}\left[\frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{4}{(2n-1)^2\pi}\cos[(2n-1)\theta]\right] = -\sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi}\sin[(2n-1)\theta],$$

which is indeed the Fourier series for $-\Pi$.

3. **Sine, cosine and half-range.**

   (a) For an odd function, we split the integral in two and make a change of variable $\theta \to -\theta$ in the second part:

$$\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L}\cos\left(\frac{\pi n\theta}{L}\right) f(\theta)\,d\theta \\
&= \frac{1}{L}\int_{0}^{L}\cos\left(\frac{\pi n\theta}{L}\right) f(\theta)\,d\theta + \frac{1}{L}\int_{-L}^{0}\cos\left(\frac{\pi n\theta}{L}\right) f(\theta)\,d\theta \\
&= \frac{1}{L}\int_{0}^{L}\cos\left(\frac{\pi n\theta}{L}\right) f(\theta)\,d\theta - \frac{1}{L}\int_{0}^{L}\cos\left(\frac{\pi n\theta}{L}\right) f(\theta)\,d\theta = 0.
\end{aligned}$$

   Note that, in the last line, we have two sign changes which cancel — one from the fact that f is odd, and the other from flipping the integration limits. This derivation works for all $n \geq 0$. The proof that b_n vanishes for even f is analogous.

(b) One approach is to work backwards. Suppose we can decompose $f = f_+ + f_-$ into odd and even parts. Then

$$f(-\theta) = f_+(-\theta) + f_-(-\theta) = f_+(\theta) - f_-(\theta).$$

We combine this with the expression for $f(\theta)$ to get

$$f_+(\theta) = \frac{1}{2}[f(\theta) + f(-\theta)], \quad f_-(\theta) = \frac{1}{2}[f(\theta) - f(-\theta)].$$

It is an easy exercise to check that these expressions are, in fact, even and odd.

(c) We can use a neat trick from first-year calculus to calculate sine and cosine coefficients at the same time. First, do the exponential integral

$$\begin{aligned}
C_n &\equiv \frac{2}{\pi}\int_0^\pi f(\theta)e^{in\theta}\,d\theta \\
&= \frac{2}{\pi}\int_0^\pi (e^\theta - 1)e^{in\theta}\,d\theta \\
&= \frac{2}{\pi}\left[\frac{e^{(in+1)\theta}}{1+in} - \frac{e^{in\theta}}{in}\right]_0^\pi \\
&= \frac{2}{\pi}\left[\frac{(-1)^n e^\pi - 1}{1+in} + i\frac{(-1)^n - 1}{n}\right] = \frac{2}{\pi}\left[\frac{(-1)^n e^\pi - 1}{1+n^2} - i(-1)^n\frac{e^\pi - 1}{n}\right].
\end{aligned}$$

Since $e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$, we immediately have $C_n = a_n + ib_n$, or

$$a_n = \frac{2[(-1)^n e^\pi - 1]}{\pi(1+n^2)}, \quad b_n = \frac{2(-1)^{n+1}(e^\pi - 1)}{\pi n}.$$

These look very different! The cosine series has coefficients $\propto 1/n^2$, and the sine series has coefficients $\propto 1/n$. Ultimately, this is due to the fact that the even extension of f is continuous, while the odd extension has a discontinuity.

4. **Dirac comb.**

(a) We can regard Ш_T as a periodic extension of $\delta(x)$ with period $T \equiv 2L$. Thus, we can represent it as a Fourier series on the domain $(-L, L]$. Using the sifting property, the coefficients of the exponential Fourier series are

$$c_n = \frac{1}{T}\int_{-L}^{L} \delta(\theta)e^{-i\omega n\theta}\,d\theta = \frac{1}{T}.$$

Thus,

$$\text{Ш}_T(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega nx} = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{i\omega nx}.$$

(b) We can split the exponential Fourier series as follows:

$$\text{Ш}_T(x) = \frac{1}{T}\sum_{n=0}^{\infty} e^{i\omega nx} + \frac{1}{T}\sum_{n=1}^{\infty} e^{-i\omega nx}.$$

If $x = Tk$ for some integer k, then

$$e^{i\omega nx} = e^{i\omega nkT} = e^{i2\pi nk} = 1 = e^{-i\omega nx}.$$

Thus, each term in the series above equals 1 and it diverges to $+\infty$. For other values of x, we can try to evaluate use a geometric series and see what happens:

$$\begin{aligned}
\text{Ш}_T(x) &= \frac{1}{T}\sum_{n=0}^{\infty} e^{i\omega nx} + \frac{1}{T}\sum_{n=1}^{\infty} e^{-i\omega nx} \\
&= \frac{1}{T}\lim_{N\to\infty}\left[\sum_{n=1}^{N-1} e^{i\omega nx} + \frac{1}{T}\sum_{n=1}^{N-1} e^{-i\omega nx}\right] \\
&= \frac{1}{T}\lim_{N\to\infty}\left[\frac{1-e^{i\omega Nx}}{1-e^{i\omega x}} + \frac{e^{-i\omega x}(1-e^{-i\omega Nx})}{1-e^{-i\omega x}}\right] \\
&= \frac{1}{T}\lim_{N\to\infty}\left[\frac{1-e^{i\omega Nx}}{1-e^{i\omega x}} - \frac{1-e^{-i\omega Nx}}{1-e^{i\omega x}}\right] \\
&= \frac{2i}{T(e^{i\omega x}-1)}\lim_{N\to\infty}\sin(\omega Nx).
\end{aligned}$$

This is not well-defined! The sine term just oscillates faster and faster as $N \to \infty$. However, there is a sense in which these infinitely oscillating functions vanish, connected to generalised functions like the Dirac delta. It is beyond the scope of the course, but look up the Riemann-Lebesgue lemma if you're interested.

5. **Numerical series.** The general idea is to pick somewhere the sine terms are easy to evaluate but nonzero. So, let's try $\theta = \pi/2$. Then $\Pi(\theta/2) = 1$, and the Fourier series is

$$\sum_{n=0}^{\infty}\frac{4}{(2n+1)\pi}\sin[(2n+1)\pi/2] = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}.$$

Equating the two gives

$$1 = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1},$$

which is equivalent to Leibniz's formula.

6. **Exponential Fourier series.** Expanding the exponential series using Euler's formula,

$$\begin{aligned}
f(\theta) &= \sum_{n=-\infty}^{\infty} c_n e^{in\omega\theta} \\
&= \sum_{n=-\infty}^{\infty} c_n\left[\cos(n\omega\theta) + i\sin(n\omega\theta)\right] \\
&= c_0 + \sum_{n=1}^{\infty}\left[(c_n + c_{-n})\cos(n\omega\theta) + i(c_n - c_{-n})\sin(n\omega\theta)\right].
\end{aligned}$$

Equating coefficients, we see that

$$a_0 = 2c_0, \quad a_n = c_n + c_{-n}, \quad b_n = i(c_n - c_{-n}).$$

Equivalently, $c_{\pm n} = (a_n \mp ib_n)/2$. It is no accident that $c_n = c^*_{-n}$; this is called the *reality condition*, and is equivalent to the Fourier series being real.

7. **Chebyshev polynomials.** We follow the instructions:

$$\begin{aligned}\int_{-1}^{1} \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}} dx &= -\int_{\pi}^{0} \frac{T_m(\cos\theta)T_n(\cos\theta)}{\sin\theta}(-\sin\theta\, d\theta) \\ &= \int_0^{\pi} \cos(n\theta)\cos(m\theta)\, d\theta.\end{aligned}$$

At this point, we remember that we have already done these integrals for Fourier series! In fact, the standard orthogonality results give

$$\int_0^{\pi} \cos(n\theta)\cos(m\theta)\, d\theta = \begin{cases} \frac{\pi}{2} & m = n \text{ and } m, n \geq 1, \\ \pi & m = n = 0, \\ 0 & \text{else.} \end{cases}$$

8. **The Gibbs phenomenon.** See `gibbs.nb`.

9. **The Basel problem.**

   (a) We pick $\theta = 0$, since both sides will be nonzero and easy to evaluate:

$$\Delta(0) = \frac{\pi}{2} + \sum_{n=0}^{\infty} \frac{4}{(2n+1)^2\pi}.$$

   Since $\Delta(0) = \pi$, we can rearrange to find

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

   (b) We separate the series B into even and odd terms:

$$\begin{aligned}B = \sum_{n=1}^{\infty} \frac{1}{n^2} &= \sum_{n=1}^{\infty} \frac{1}{(2n)^2} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \\ &= \sum_{n=1}^{\infty} \frac{1}{(2n)^2} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{1}{4}B + \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.\end{aligned}$$

   Hence,

$$\frac{3}{4}B = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \quad \Longrightarrow \quad B = \frac{4}{3}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{6}.$$

10. **Harmonic symmetry.***

(a) We simply compute:

$$\begin{aligned} f_j(\omega^m z) &= \frac{1}{n}\sum_{k=0}^{n-1} f(\omega^{k+m}z)\omega^{-jk} \\ &= \frac{1}{n}\sum_{k=0}^{n-1} f(\omega^{k}z)\omega^{-j(k-m)} \\ &= \omega^{jm}\frac{1}{n}\sum_{k=0}^{n-1} f(\omega^{k}z)\omega^{-jk} = \omega^{jm} f_j(z). \end{aligned}$$

On the second line, we relabelled the dummy index $k \to k+m$, using the n-fold symmetry of the sum over roots of unity. Similarly, we can swap the order of the finite sums over j and k, and use the formula for geometric sums, to get

$$\begin{aligned} \sum_{j=0}^{n-1} f_j(z) &= \sum_{j=0}^{n-1}\frac{1}{n}\sum_{k=0}^{n-1} f(\omega^k z)\omega^{-jk} \\ &= \frac{1}{n}\sum_{k=0}^{n-1} f(\omega^k z)\sum_{j=0}^{n-1}\omega^{-jk} \\ &= \frac{1}{n}\cdot nf(z) + \frac{1}{n}\sum_{k=1}^{n-1} f(\omega^k z)\frac{1-\omega^{-kn}}{1-\omega^{-k}} = f(z). \end{aligned}$$

In the last step, we used the fact that $\omega^{-kn} = (\omega^n)^{-k} = 1$. Hence,

$$f(z) = \sum_{j=0}^{n-1} f_j(z).$$

(b) Setting $n = 2$, we note that the second roots of unity are just ± 1. The results in (a) become

$$f_\pm(z) = \frac{1}{2}(f(z) \pm f(-z)), \quad f(z) = f_+(z) + f_-(z).$$

This is exactly what we found (albeit for real functions) in Problem 3(b).

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Wave Theory and Fourier Analysis: Tutorial 2

Tutorial problems

1. **Series and differentiation.** Sometimes, we can take a function defined in terms of a series, differentiate it with respect to a parameter, and end up with a new and useful result. In this context, we won't worry too much about convergence. Let's see an example! In a later problem, you can use the results of your calculation to find the "sum" of all natural numbers.

   (a) Using the geometric series, show that for $\alpha > 0$,

   $$\sum_{n=0}^{\infty} e^{-n\alpha} = \frac{1}{1 - e^{-\alpha}}.$$

   (b) Differentiating both sides with respect to α, derive the identity

   $$\sum_{n=1}^{\infty} n e^{-n\alpha} = \frac{e^{\alpha}}{(e^{\alpha} - 1)^2}. \tag{1}$$

2. **Series solution of ODEs.** Series give us a powerful method for solving ordinary differential equations. Let's try an ODE we can check using elementary methods:

   $$y'' + \omega^2 y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

   This has solution $y = \cos(\omega x)$. Arrive at the same solution using a *power series*

   $$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

   In this question, you may assume power series are unique. You will also need the Taylor series for cosine,

   $$\cos(\omega x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (\omega x)^{2n}.$$

3. **D'Alembert's formula.** Verify that

   $$u(x,t) = \frac{1}{2}\big[f(x - vt) + f(x + vt)\big] + \frac{1}{2v} \int_{x-vt}^{x+vt} g(y)\, dy$$

   solves the initial value problem for the wave equation:

   $$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2}, \quad u(x,0) = f(x), \quad \dot{u}(x,0) = g(x).$$

4. **Tuning a clarinet.** In acoustic applications of Fourier series, periodic functions represent *sound waves*, i.e. fluctuations around atmospheric pressure. Sine and cosine terms are called *pure tones*; squaring the coefficient tells you how loud the tone is. It turns out we can learn a lot about instruments just using first-year physics and Fourier series.

(a) For a half-open pipe of length A (e.g. a clarinet), what are the allowed wavelengths? Convert them into frequencies using the speed of sound v_s. You should find allowed angular frequencies

$$\omega_n = \frac{\pi v_s(2n+1)}{2A}, \quad n = 0, 1, 2, \ldots$$

These are the *odd* harmonics; even harmonics can't resonate and quickly die away.

(b) Suppose we force the air in the clarinet to vibrate at a *non-resonant* pure tone, with waveform $\sin(2\pi f\theta)$. Calculate the Fourier series over the basic interval $[-L, L]$, where $L \equiv 2A/v_s$. You should find $a_n = 0$ and

$$b_n = \frac{2nL(-1)^n \sin(2\pi fL)}{\pi[(2fL)^2 - n^2]}.$$

Which harmonic do you expect to be loudest?

5. **Legendre polynomials.** The *Legendre polynomials* P_n can be defined by *Rodrigues' formula*:

$$P_n(x) \equiv \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$$

They form an orthogonal basis for functions on $[-1, 1]$, with respect to the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\, dx.$$

(a) Use Rodrigues' formula to calculate the first three Legendre polynomials. Apply the Gram-Schmidt procedure to the ordered basis $(1, x, x^2)$ and show that the two methods agree up to normalisation. You can generate all Legendre polynomials from Gram-Schmidt, but Rodrigues' formula is much nicer.

(b) Using the Rodrigues' formula and integration by parts n times, show that

$$\int_{-1}^{1} P_n(x)P_m(x)\, dx \propto \int_{-1}^{1} (-1)^n(x^2-1)^n \cdot D^{m+n}(x^2-1)^m\, dx$$

where $D \equiv d/dx$ is the derivative operator. Conclude that the P_n are orthogonal.

Extra problems

6. **Peak hour diffusion.** At peak hour ($t = 0$), commuters cram onto the train and begin the long trek home. Each carriage has length T and a single door halfway down the carriage. If there are N commuters, we can model the initial commuter density distribution as

$$n(x, 0) = N\delta\big(x - \tfrac{1}{2}T\big).$$

Since commuters like their own space, we can model the attraction to regions of lower density with a *diffusion equation*:

$$\frac{\partial}{\partial t} n(x,t) = D\frac{\partial^2}{\partial x^2} n(x,t), \quad \frac{\partial}{\partial x} n(0,t) = \frac{\partial}{\partial x} n(T,t) = 0. \tag{2}$$

The boundary conditions ensure that no commuters "leak out" of the ends of the carriage.

(a) Show that a separable solution $n(x,t) = \phi(x)e^{-\lambda t}$, $\lambda \geq 0$, must satisfy

$$\frac{d^2\phi}{dx^2} = -\omega^2\phi, \quad \omega^2 \equiv \frac{\lambda}{D}. \tag{3}$$

(b) Solve (3), and verify that the boundary conditions for $n(x,t)$ imply

$$\omega T = k\pi \tag{4}$$

for some integer k. Label the corresponding choices of ω, λ by ω_k, λ_k.

(c) Using the linearity of (2), combine the results of (a) and (b) to obtain

$$n(x,t) = \sum_{k=0}^{\infty} a_k \cos(\omega_k x)\, e^{-\lambda_k t}.$$

From the Fourier series for $n(x, 0)$ (or orthogonality directly), verify the coefficients

$$a_0 = \frac{N}{T}, \quad a_k = \frac{2N}{T}\cos\left(\frac{\omega_k T}{2}\right), \quad k > 0.$$

(d) As $t \to \infty$, what does $n(x,t)$ look like? Is the total number of commuters conserved?

7. **Epicycles.** Recall that we can write the Fourier series for f in terms of exponentials,

$$f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega n\theta}, \quad c_n = \frac{1}{2L}\int_{-L}^{L} f(\theta)e^{-i\omega n\theta}\, d\theta, \quad \omega = \frac{\pi}{L}.$$

This remains true for functions which execute periodic motion in the complex plane.

Interpret each term of the form $c_n e^{i\omega n\theta}$ geometrically. Deduce that any periodic motion on the plane can be decomposed into a sum of *circular motions*. In light of this observation, why is Ptolemy's theory of celestial motion unfalsifiable for periodic orbits?

8. **The method of Frobenius.** Here's a trickier ODE we can solve with power series:

$$x^2y'' + 4xy' + (x^2 + 2)y = 0.$$

We try *the method of Frobenius*, which uses a generalised power series of the form

$$y(x) = x^s \sum_{n=0}^{\infty} a_n x^n \tag{5}$$

where s is some rational number to be determined, and $a_0 \neq 0$ (otherwise we change the definition of s). Show that $s = -2$, $s = -1$ are consistent, and for $s = -2$, derive the solution

$$y = \frac{a_0 \sin x}{x^2}.$$

You will need the Taylor series for sine,

$$\sin x = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!}.$$

9. **Fundamental solutions of Laplace's equation.*** NOTE: *This problem requires some vector calculus; ignore it if you haven't done the subject!* For three spatial dimensions, verify that

$$\Phi(\mathbf{x}) = -\frac{1}{4\pi|\mathbf{x}|}$$

solves the inhomogeneous Laplace equation

$$\nabla^2\Phi(\mathbf{x}) = -\delta(\mathbf{x}).$$

HINT. Let B be the unit ball and $S = \partial B$ the unit sphere. Apply Gauss' theorem:

$$\int_B \nabla \cdot \mathbf{F}\, dV = \oint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dA.$$

Recall that $\hat{\mathbf{n}}$ is an outward pointing unit normal.

10. **An infamous sum.*** Expand the RHS of (1) as a Taylor series in α to find that

$$\sum_{n=1}^{\infty} ne^{-n\alpha} = \frac{1}{\alpha^2} - \frac{1}{12} + \text{higher order terms.}$$

In some physical contexts (e.g. the Casimir effect), we can take $\alpha \to 0$ and throw away the α^{-2} term on physical grounds, yielding the somewhat infamous equation

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + \cdots \overset{?!}{=} -\frac{1}{12}.$$

Here are some results on power series you may find useful:

$$e^{\alpha} = 1 + \alpha + \frac{1}{2}\alpha^2 + \frac{1}{6}\alpha^2 + \cdots$$

$$\frac{1}{(1 + a_1x + a_2x^2 + \cdots)^2} = 1 - 2a_1x + (3a_1^2 - 2a_2)x^2 + \cdots .$$

11. **Hermite polynomials.*** The Hermite polynomials $H_n(x)$, $n = 0, 1, 2, \ldots$, are a family of polynomials on $\mathbb{R}$, orthogonal and complete with respect to the inner product

$$\langle f, g \rangle \equiv \int_{-\infty}^{\infty} e^{-x^2} f(x) g(x)\, dx.$$

In physics, they are most famous as the wavefunctions of the quantum harmonic oscillator. In this problem, we will get familiar with the *generating function* for the Hermite polynomials,

$$G(s, x) \equiv e^{2sx - s^2} \equiv \sum_{n=0}^{\infty} H_n(x) \frac{s^n}{n!}.$$

In other words, we start with a function $G(s, x)$ of two variables, expand it as a Taylor series in s, and *define* $H_n(x)/n!$ as the coefficient of s^n.

(a) Using the definition of the Taylor expansion, derive *Rodrigues' formula* for Hermite polynomials:

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

Explain briefly why this is a polynomial.

(b) By calculating the partial derivative of G with respect to x in two ways, derive the recurrence relation

$$H'_{n+1}(x) = 2(n+1) H_n(x), \quad n \geq 0. \tag{6}$$

Similarly, from the partial derivative with respect to s, deduce the recurrence relation

$$H_{n+1}(x) - 2x H_n(x) + 2n H_{n-1}(x) = 0, \quad n \geq 1. \tag{7}$$

(c) Combine the results in (b) to yield the differential equation satisfied by the Hermite polynomials,

$$H''_n(x) - 2x H'_n(x) + 2n H_n(x) = 0.$$

PHYC20014 Physical Systems

Wave Theory and Fourier Analysis: Tutorial 2

Solutions

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1. **Series and differentiation.**

   (a) Now is a good time to brush off the cobewbs and recall the geometric series:

   $$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad \text{provided } |x| < 1.$$

   Since $\alpha > 0$, $|e^{-\alpha}| < 1$ and we can use the above result:

   $$\sum_{n=0}^{\infty} e^{-n\alpha} = \sum_{n=0}^{\infty} (e^{-\alpha})^n = \frac{1}{1-e^{-\alpha}}.$$

   (b) Let's differentiate with respect to α, assuming we can differentiate term-by-term. For the power series on the LHS, this gives

   $$\frac{d}{d\alpha}\sum_{n=0}^{\infty} e^{-n\alpha} = \sum_{n=0}^{\infty} \frac{d}{d\alpha} e^{-n\alpha} = -\sum_{n=1}^{\infty} n e^{-n\alpha}.$$

   For the expression on the RHS, we get

   $$\frac{d}{d\alpha}\frac{1}{1-e^{-\alpha}} = \frac{-e^{-\alpha}}{(1-e^{-\alpha})^2} = -\frac{e^{\alpha}}{(e^{\alpha}-1)^2}.$$

   Equating the two gives (1). (If you are interested, the property of the series that allows us to differentiate term-by-term is called *uniform convergence*.)

2. **Series solution of ODEs.** So, we guess a power series solution of the form

   $$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

   The initial condition $y(0) = 1$ implies $a_0 = 1$, while $y'(0) = 0$ implies $a_1 = 0$. Differentiating,

   $$y''(x) = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} = \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n.$$

   Thus, our original ODE becomes

   $$y''(x) + \omega^2 y(x) = \sum_{n=0}^{\infty} \Big[a_{n+2}(n+2)(n+1) + \omega^2 a_n\Big] x^n = 0.$$

From the uniqueness of power series, the coefficients vanish, i.e.

$$a_{n+2}(n+2)(n+1) + \omega^2 a_n = 0.$$

Since $a_1 = 0$, it follows immediately that for all *odd* n, $a_n = 0$. For even n, we have

$$a_0 = 1 \quad \implies \quad a_2 = -\frac{\omega^2 a_0}{1 \cdot 2} = (-1)^1 \frac{\omega^2}{2!} \quad \implies \quad a_4 = -\frac{\omega^2 a_2}{3 \cdot 4} = (-1)^2 \frac{\omega^4}{4!} \quad \cdots$$

Hopefully you can see the pattern. Otherwise, you can use induction to show that

$$a_{2n} = (-1)^n \frac{\omega^{2n}}{(2n)!}.$$

Hence, our power series solution is

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (-1)^n \frac{(\omega x)^{2n}}{(2n)!} = \cos(\omega x).$$

Of course, this is much more work than the elementary solution. Unlike elementary methods, however, the series solution approach is very general and powerful.

3. **D'Alembert's formula.** Since all terms in $u(x,t)$ are a function of $x \pm vt$, it clearly solves the wave equation. At $t = 0$,

$$u(x,0) = \frac{1}{2}\big[f(x) + f(x)\big] + \frac{1}{2v}\int_x^x g(y)\,dy = f(x)$$

so the first initial condition is satisfied. For the second, note that

$$\begin{aligned} \dot{u}(x,t) &= \frac{1}{2}\big[-vf(x-vt) + vf(x+vt)\big] + \frac{1}{2v}\big[vg(x+vt) - vg(x-vt)\big] \\ \implies \qquad \dot{u}(x,0) &= g(x). \end{aligned}$$

Thus, $u(x,t)$ solves the initial value problem.

4. **Tuning a clarinet.**

(a) Since a half-open pipe has a node at one end and an anti-node at the other, the allowed wavelengths are

$$\lambda_n = \frac{4A}{2n+1}, \qquad n = 0, 1, 2, \ldots$$

Since $v_s = \lambda_n \omega_n / 2\pi$, the allowed angular frequencies are

$$\omega_n = \frac{2\pi v_s}{\lambda_n} = \frac{\pi v_s (2n+1)}{A}, \qquad n = 0, 1, 2, \ldots$$

(b) The non-resonant pure tone $\sin(2\pi f\theta)$ is an *odd* function, so the a_n terms vanish. Using a sine series and some trig identities, we get

$$\begin{aligned}
b_n &= \frac{2}{L}\int_0^L \sin\left(\frac{n\pi\theta}{L}\right)\sin(2\pi f)\,d\theta \\
&= \frac{1}{L}\int_0^L \left\{\cos\left[\left(\frac{n\pi}{L}-2\pi f\right)\theta\right]-\cos\left[\left(\frac{n\pi}{L}+2\pi f\right)\theta\right]\right\}d\theta \\
&= \frac{1}{L}\left[\frac{\sin[\left(\frac{n\pi}{L}-2\pi f\right)\theta]}{n\pi/L-2\pi f}-\frac{\sin[\left(\frac{n\pi}{L}+2\pi f\right)\theta]}{n\pi/L+2\pi f}\right]_0^L \\
&= \frac{1}{L}\cdot\frac{2n\pi\sin(2\pi fL)\cos(n\pi)}{(2\pi f)^2-(n\pi/L)^2} \\
&= \frac{2nL(-1)^n\sin(2\pi fL)}{\pi[(2fL)^2-n^2]}.
\end{aligned}$$

Generally, b_n will peak at the *odd* n where the denominator is smallest, that is, the odd n closest to $2fL$.

5. **Legendre polynomials.**

(a) Let's start with Rodrigues' formula:

$$\begin{aligned}
P_0(x) &\equiv (x^2-1)^0 = 1 \\
P_1(x) &\equiv \frac{1}{2}\frac{d}{dx}(x^2-1) = x \\
P_2(x) &\equiv \frac{1}{8}\frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{2}\frac{d}{dx}(x^3-x) = \frac{1}{2}(3x^2-1).
\end{aligned}$$

Now we apply Gram-Schmidt to $(1,x,x^2)\equiv(f_1,f_2,f_3)$ to yield the orthonormal set (g_1,g_2,g_3). Remember that the general idea is to step through the original basis vectors, projecting out the part orthogonal to the orthonormal basis you've constructed so far, then normalise what's left. We label these vectors: $f\to f'\to g$.

Since we start with an empty orthogonal basis, the first step is just to normalise f_1:

$$|f_1|^2 = \int_{-1}^1 dx = 2 \quad\Longrightarrow\quad g_1(x) = \frac{1}{\sqrt{2}} \propto P_0(x).$$

Now we project out the g_1 dependence in f_2 to get f_2':

$$\langle f_2, g_1\rangle = \int_{-1}^1 \frac{1}{\sqrt{2}}x\,dx = 0 \quad\Longrightarrow\quad f_2' = f_2 - \langle f_2,g_1\rangle g_1 = f_2.$$

Now we normalise $f_2' = f_2$:

$$|f_2|^2 = \int_{-1}^1 x^2\,dx = \frac{2}{3} \quad\Longrightarrow\quad g_2(x) = \sqrt{\frac{3}{2}}x \propto P_1(x).$$

Finally, we project out the g_1 and g_2 dependence in f_3:

$$\begin{aligned}\langle f_3, g_1\rangle &= \int_{-1}^{1} \frac{1}{\sqrt{2}} x^2\, dx = \frac{\sqrt{2}}{3} \\ \langle f_3, g_2\rangle &= \int_{-1}^{1} \sqrt{\frac{3}{2}} x^3\, dx = 0 \\ \implies \qquad f_3'(x) &= f_3(x) - \langle f_3, g_1\rangle g_1(x) - \langle f_3, g_2\rangle g_2(x) = x^2 - \frac{1}{3} \propto P_3(x).\end{aligned}$$

We note that $g_3 \propto f_3'$, and hence $g_3 \propto P_3$. Thus, Rodrigues' formula agrees with Gram-Schmidt up to normalisation.

(b) We don't need to worry too much about normalisation here, since everything will vanish. Consider the inner product of two Legendre polynomials P_n and P_m, and assume without loss of generality that $n \geq m$:

$$\langle P_n, P_m\rangle = \int_{-1}^{1} P_n(x) P_m(x)\, dx \propto \int_{-1}^{1} D^n(x^2-1)^n \cdot D^m(x^2-1)^m.$$

Applying integration by parts once gives

$$\left[D^{n-1}(x^2-1)^n \cdot D^{m+1}(x^2-1)^m\right]_{-1}^{1} - \int_{-1}^{1} D^{n-1}(x^2-1)^n \cdot D^{m+1}(x^2-1)^m.$$

In the first term, applying the derivative operator D^{n-1} to the polynomial $(x^2-1)^n$ leaves an overall factors of (x^2-1) out the front (using the chain rule). Hence, the surface terms vanish, and we conclude

$$\int_{-1}^{1} D^n(x^2-1)^n \cdot D^m(x^2-1)^m = -\int_{-1}^{1} D^{n-1}(x^2-1)^n \cdot D^{m+1}(x^2-1)^m.$$

Applying the same trick n times, we obtain

$$\langle P_n, P_m\rangle = \int_{-1}^{1} (-1)^n (x^2-1)^n \cdot D^{m+n}(x^2-1)^m\, dx.$$

Finally, if we assume the polynomials are distinct, with $n \neq m$, then our initial assumption becomes $n > m$. It follows that $n+m > 2m$, and the derivative kills the polynomial, which is of order $2m$:

$$D^{n+m}(x^2-1)^m = 0.$$

Hence, Legendre polynomials are orthogonal:

$$\langle P_n, P_m\rangle = 0, \quad n \neq m.$$

6. **Peak hour diffusion.**

(a) For a separable solution $n(x,t) = \phi(x)e^{-\lambda t}$, the diffusion equation (2) implies

$$-\lambda\phi(x)e^{-\lambda t} = D\phi''(x)e^{-\lambda t} \quad \implies \frac{d^2\phi}{dx^2} = -\frac{\lambda}{D}\phi.$$

(b) For $\omega > 0$, the solutions to (3) are just trigonometric functions:

$$\phi(x) = a\cos(\omega x) + b\sin(\omega x).$$

However, the boundary conditions $n'(0,t) = n'(T,t) = 0$ imply $\phi'(0) = \phi'(T) = 0$, or

$$\begin{aligned}\phi'(0) &= \omega[-a\sin(\omega\cdot 0) + b\cos(\omega\cdot 0)] = -\omega B = 0 \\ \phi'(T) &= -\omega a\sin(\omega T) = 0,\end{aligned}$$

where we have simplified the second equation using the first. The second implies $\omega T = k\pi$ for some integer $k \neq 0$. For $\omega = 0$, the equation $\phi'' = 0$ has general solution $\phi(x) = ax+b$; the boundary conditions imply $a = 0$, so we end up with the constant solution.

(c) We have a different solution for each nonnegative integer value of k. Since the diffusion equation is linear, we can combine them to get a general solution

$$n(x,t) = \sum_{k=0}^{\infty}\phi_k(x)e^{-\lambda_k t} = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\cos(\omega_k x)\, e^{-\lambda_k t}.$$

Recall our initial condition

$$n(x,0) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\cos(\omega_k x) = N\delta(x - \tfrac{1}{2}T).$$

Using orthogonality directly (and sneakily including the constant term in $k = 0$),

$$\begin{aligned}a_k &= \frac{2}{T}\int_{-1}^{1} n(x,0)\cos(\omega_k x)\ dx \\ &= \frac{2}{T}\int_{-1}^{1} N\delta(x-\tfrac{1}{2}T)\cos(\omega_k x)\ dx = \frac{2N}{T}\cos\left(\frac{\omega_k T}{2}\right).\end{aligned}$$

(d) Separable terms $\phi_k(x)e^{-\lambda_k t}$ with $\lambda_k > 0$ decay exponentially; as $t \to \infty$, they vanish. Since $\lambda_0 = 0$, only the $k = 0$ term survives. Hence,

$$n(x,t) \sim \frac{a_0}{2} = \frac{N}{T}.$$

This is a constant! Integrating over the length of the carriage, we see that the total number of commuters is indeed conserved:

$$\int_0^T n(x,\infty)\, dx = \int_0^T \frac{N}{T}\, dx = N.$$

7. **Epicycles.** Geometrically, each term $c_n e^{i\omega n\theta}$ describes motion in a circle centred at the origin. Writing $c_n = re^{i\delta}$, the circular motion has

- angular speed $\omega|n|$,
- direction given by the sign of n (+ccw and −cw),
- radius r,
- phase offset δ.

Since Ptolemy's theory of epicycles breaks periodic orbits into a sum of a such circular motions, it has no physical content: it is just the exponential Fourier series!

8. **The method of Frobenius.** The approach is very similar to Problem 2. The generalised power series, and its derivatives, are

$$
\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^{n+s} \\
y'(x) &= \sum_{n=0}^{\infty} a_n (n+s) x^{n+s-1} \\
y''(x) &= \sum_{n=0}^{\infty} a_n (n+s)(n+s-1) x^{n+s-2}.
\end{aligned}
$$

The LHS of our ODE becomes

$$
\begin{aligned}
x^2 y'' + 4xy' + (x^2+2)y &= \sum_{n=0}^{\infty} a_n (n+s)(n+s-1) x^{n+s} + \sum_{n=0}^{\infty} 4a_n (n+s) x^{n+s} + \sum_{n=0}^{\infty} (x^2+2) a_n x^{n+s} \\
&= \sum_{n=0}^{\infty} \left[\left\{ (n+s)(n+s-1) + 4(n+s) + 2 \right\} a_n + a_{n-2} \right] x^{n+s}
\end{aligned}
$$

where on the last line, we extended the lower bound on the sum from $n = 0$ to $n = -2$ by setting $a_{-2} = a_{-1} = 0$. Setting the coefficients to zero, we obtain the recurrence relation

$$
\begin{aligned}
0 &= \left\{ (n+s)(n+s-1) + 4(n+s) + 2 \right\} a_n + a_{n-2} \\
&= \left\{ (n+s)(3+n+s) + 2 \right\} a_n + a_{n-2}.
\end{aligned}
$$

Consider the $n = 0$ term:

$$
(s^2 + 3s + 2) a_0 = (s+1)(s+2) a_0 = 0.
$$

Since $a_0 \neq 0$, this implies $s = -1$ or $s = -2$. Selecting $s = -1$, the recurrence becomes

$$
a_n = -\frac{a_{n-2}}{n(n+1)}.
$$

As in Problem 2, by computing some examples, or using induction, you can show that $a_{2n+1} = 0$, and

$$
a_{2n} = \frac{(-1)^n a_0}{(2n+1)!}.
$$

The corresponding generalised power series is

$$
y(x) = \sum_{n=0}^{\infty} a_n x^{n+s} = \sum_{n=0}^{\infty} \frac{(-1)^n a_0}{(2n+1)!} x^{2n-1} = \frac{a_0 \sin x}{x^2}.
$$

This was only slightly more work than solving the trivial ODE in Problem 2! In general, series solutions "scale" well. More often than not, the issue is not finding the recurrence relation, but figuring out if your power series can be written in a nicer way.

9. **Fundamental solutions of Laplace's equation.** To verify that $\nabla^2\Phi(\mathbf{x}) = -\delta(\mathbf{x})$, we first check that

$$\nabla^2\Phi(\mathbf{x}) = 0, \quad \mathbf{x} \neq \mathbf{0}.$$

Since $|\mathbf{x}|^{-1} = (x^2+y^2+z^2)^{-1/2}$, you can easily compute

$$\nabla\left(\frac{1}{|\mathbf{x}|}\right) = -\frac{\mathbf{x}}{|\mathbf{x}|^3}, \quad \nabla\left(\frac{1}{|\mathbf{x}|^3}\right) = -\frac{3\mathbf{x}}{|\mathbf{x}|^5}.$$

Hence, using vector identity 6 of Appendix A of the notes, for $\mathbf{x} \neq \mathbf{0}$:

$$\begin{aligned}
\nabla^2\Phi(\mathbf{x}) &= -\frac{1}{4\pi}\nabla\cdot\nabla\left(\frac{1}{|\mathbf{x}|}\right) \\
&= \frac{1}{4\pi}\nabla\cdot\left(\frac{\mathbf{x}}{|\mathbf{x}|^3}\right) \\
&= \frac{1}{4\pi}\left[\frac{1}{|\mathbf{x}|^3}(\nabla\cdot\mathbf{x}) + \mathbf{x}\cdot\nabla\left(\frac{1}{|\mathbf{x}|^3}\right)\right] \\
&= \frac{1}{4\pi}\left[\frac{3}{|\mathbf{x}|^3} - \frac{3\mathbf{x}\cdot\mathbf{x}}{|\mathbf{x}|^5}\right] = \frac{1}{4\pi}\left[\frac{3}{|\mathbf{x}|^3} - \frac{3}{|\mathbf{x}|^3}\right] = 0.
\end{aligned}$$

Figuring out the behaviour at $\mathbf{x} = \mathbf{0}$ is a bit trickier, but the main thing is to reproduce the sifting property

$$\int_V \nabla^2\Phi(\mathbf{x}) = -1,$$

where V is any volume containing the origin. Let B be the unit ball. By Gauss' theorem,

$$\int_B \nabla\cdot\nabla\Phi(\mathbf{x}) = \oint_S \nabla\Phi(\mathbf{x})\cdot\hat{\mathbf{n}}\,dA = \frac{1}{4\pi}\oint_S \frac{\mathbf{x}\cdot\mathbf{x}}{|\mathbf{x}|^3}\,dA = \frac{1}{4\pi}\int_S dA = 1,$$

where we have used the fact that $\hat{\mathbf{n}} = \mathbf{x}$, $|\mathbf{x}| = 1$ on the unit sphere.

10. **An infamous sum.*** Using (1) and the supplied power series identities,

$$\begin{aligned}
\sum_{n=1}^{\infty} ne^{-n\alpha} &= \frac{e^\alpha}{(e^\alpha-1)^2} \\
&= e^\alpha\cdot\frac{1}{(e^\alpha-1)^2} \\
&= \left(1+\alpha+\frac{1}{2}\alpha^2+\cdots\right)\frac{1}{\alpha^2(1+\frac{1}{2}\alpha+\frac{1}{6}\alpha^2+\cdots)^2} \\
&= \frac{1}{\alpha^2}\left(1+\alpha+\frac{1}{2}\alpha^2+\cdots\right)\left(1-\alpha+\left[\frac{3}{4}-\frac{2}{6}\right]\alpha^2+\cdots\right) \\
&= \frac{1}{\alpha^2}\left(1+\left[\frac{1}{2}-1+\frac{5}{12}\right]\alpha^2+\cdots\right) \\
&= \frac{1}{\alpha^2}-\frac{1}{12}+\cdots.
\end{aligned}$$

Note that, at each point, we can tell how many terms to keep in the power series expansions by thinking about how many terms we require in our final result.

11. **Hermite polynomials.***

(a) Recall that in a Taylor series, the coefficients are related to derivatives of f:

$$f(s) = \sum_{n=0}^{\infty} \frac{s^n}{n!} \frac{d^n f(s)}{ds^n}\bigg|_{s=0}.$$

By definition, the $H_n(x)$ are Taylor coefficients of $G(s, x)$, so

$$H_n(x) = \frac{d^n G(s,x)}{ds^n}\bigg|_{s=0}.$$

Noting that

$$G(s,x) = e^{2sx-s^2} = e^{-x^2} e^{-(s-x)^2},$$

it follows that

$$\begin{aligned} H_n(x) &= \frac{d^n G(s,x)}{ds^n}\bigg|_{s=0} \\ &= e^{-x^2} \frac{d^n}{ds^n} e^{-(s-x)^2}\bigg|_{s=0} \\ &= e^{-x^2} \frac{d^n}{dt^n} e^{-t^2}\bigg|_{t=-x} \\ &= (-1)^n e^{-x^2} \frac{d^n}{dx^n} e^{-x^2}. \end{aligned}$$

Each derivative just brings down polynomials by the chain rule, but leaves an overall factor e^{-x^2} to be cancelled by the e^{x^2} prefactor.

(b) Let's calculate the partial derivative with respect to x, first for the closed form of $G(s, x)$:

$$\frac{\partial}{\partial x} e^{2sx-s^2} = 2sG(s,x) = \sum_{n=0}^{\infty} 2H_n(x) \frac{s^{n+1}}{n!}.$$

Now let's differentiate the power series directly:

$$\frac{\partial}{\partial x} \sum_{n=0}^{\infty} H_n(x) \frac{s^n}{n!} = \sum_{n=1}^{\infty} H_n'(x) \frac{s^n}{n!} = \sum_{n=0}^{\infty} H_{n+1}'(x) \frac{s^{n+1}}{n!(n+1)},$$

using $H_0(x)' = (1)' = 0$ in the first equality. Equating the s^n coefficients yields

$$H_{n+1}'(x) = 2(n+1)H_n(x), \quad n \geq 0.$$

Similarly, for the partial derivative for s, we have

$$\begin{aligned} \frac{\partial}{\partial s} e^{2sx-s^2} &= 2(x-s)G(s,x) \\ &= \sum_{n=0}^{\infty} 2(x-s)H_n(x) \frac{s^n}{n!} \\ &= \sum_{n=0}^{\infty} 2xH_n(x) \frac{s^n}{n!} - \sum_{n=1}^{\infty} 2H_{n-1}(x) \frac{s^n}{n!} \end{aligned}$$

and

$$\frac{\partial}{\partial s}\sum_{n=0}^{\infty} H_n(x)\frac{s^n}{n!} = \sum_{n=1}^{\infty} H_n(x)\frac{s^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} H_{n+1}(x)\frac{s^n}{n!}.$$

Equating the two and shuffling terms around, we obtain

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0, \quad n \geq 1.$$

(c) We use (6) to simplify (7), writing everything in terms of H_{n+1}:

$$\begin{aligned} 0 &= H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) \\ &= H_{n+1}(x) - \frac{2x}{2(n+1)}H'_{n+1}(x) + \frac{1}{2(n+1)}H''_{n+1}(x). \end{aligned}$$

Multiplying through by $2(n+1)$ and shifting $n+1 \to n$, we obtain

$$H''_n(x) - 2xH'_n(x) + 2nH_n(x) = 0.$$

PHYC20014 Physical Systems

Wave Theory and Fourier Analysis: Tutorial 3

Tutorial problems

1. **Basic Fourier transforms.** Fourier transform the following functions. NOTE. Our convention for Fourier transforms and inverses is
$$\hat{\mathcal{F}}[f](u) = \int_{-\infty}^{\infty} f(x)e^{-2\pi iux}\,dx, \quad \hat{\mathcal{F}}^{-1}[F](x) = \int_{-\infty}^{\infty} F(u)e^{2\pi iux}\,du.$$
   (a) The *rectangle* function rect, defined by
$$\operatorname{rect}(x) = \begin{cases} 1 & |x| < \frac{1}{2} \\ \frac{1}{2} & |x| = \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$
   (b) The *triangle* function Λ, defined by
$$\Lambda(x) = \begin{cases} 1 - |x| & |x| < 1 \\ 0 & \text{otherwise.} \end{cases}$$

2. **Fourier transform properties.** From basic Fourier transforms, we move on to results *about* Fourier transforms. As usual, F denotes the Fourier transform $\hat{\mathcal{F}}[f(x)]$. Prove the following:
   (a) *Similarity.* For any constant $a \neq 0$,
$$\hat{\mathcal{F}}[f(ax)] = \frac{1}{|a|}F\left(\frac{u}{a}\right).$$
   (b) *Shifts.* For any constant a,
$$\hat{\mathcal{F}}[f(x-a)] = F(u)e^{-2\pi iua}.$$
   (c) *Derivatives.* Assuming f vanishes as $x \to \pm\infty$,
$$\hat{\mathcal{F}}[f'(x)] = 2\pi iuF(u).$$

3. **Gaussians.** A *Gaussian* is a function of the form $\exp(-\alpha x^2)$; they are of supreme importance to physics, both in theory and practice. We explore a few of their basic properties.
   (a) Let
$$I(\alpha) \equiv \int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx.$$
   By changing from Cartesian to polar coordinates, prove that
$$I(\alpha)^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\alpha(x^2+y^2)}\,dx\,dy = \int_0^{\infty}\int_0^{2\pi} e^{-\alpha r^2}(r\,d\phi\,dr).$$
   Deduce the famous *Gaussian integral*:
$$I(\alpha) = \sqrt{\frac{\pi}{\alpha}}. \tag{1}$$

(b) Complete the square in the exponent to show that

$$\int_{-\infty}^{\infty} e^{-\alpha x^2+\beta x}\,dx = \sqrt{\frac{\pi}{\alpha}}\exp\left(\frac{\beta^2}{4\alpha}\right). \tag{2}$$

Use this result[1] to compute

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} e^{i\beta x^2} e^{-2\pi i u x}\,dx.$$

This integral crops up when calculating diffraction patterns in optics.

(c) Verify that the Fourier transform of a Gaussian is another Gaussian, of the form

$$\hat{\mathcal{F}}[e^{-\alpha x^2}](u) = \sqrt{\frac{\pi}{\alpha}}\exp\left(-\frac{\pi^2 u^2}{\alpha}\right). \tag{3}$$

4. **Convolutions.** Recall that the convolution of f and g is defined by

$$(f * g)(x) \equiv \int_{-\infty}^{\infty} g(\xi) f(x-\xi)\,d\xi = \int_{-\infty}^{\infty} f(\xi) g(x-\xi)\,d\xi.$$

(a) Prove the *convolution theorem*:

$$\hat{\mathcal{F}}[f * g] = \hat{\mathcal{F}}[f]\hat{\mathcal{F}}[g].$$

HINT. Use the sneaky factorisation $e^{-2\pi i u x} = e^{-2\pi i u \xi} e^{-2\pi i u (x-\xi)}$.

(b) Sketch the convolution rect $*$ rect *without* doing any calculation. What is it?

5. **Heaviside step and Dirac delta.** The *Heaviside step function* $\Theta : \mathbb{R} \to \mathbb{R}$ models a signal instantaneously switching on at $x = 0$:

$$\Theta(x) = \begin{cases} +1 & x > 0 \\ \frac{1}{2} & x = 0 \\ 0 & x < 0. \end{cases}$$

(a) Show that, in the sense of generalised functions, $\Theta'(x) = \delta(x)$. HINT. Integrate $\Theta'(x)$ against a test function and use integration by parts. Alternatively, integrate directly.

(b) Using $\hat{\mathcal{F}}[\delta(x)] = 1$ and $\hat{\mathcal{F}}[f'(x)] = 2\pi i u F(u)$, derive the Fourier transform

$$\hat{\mathcal{F}}[\Theta(x)] = \frac{1}{2\pi i u}.$$

[1] You may assume (2) holds for *complex* numbers β and α, with $\mathrm{Re}(\alpha) > 0$.

Extra problems

6. **More fun with Fourier transforms.** Assorted exercises on Fourier transforms.

   (a) Show that $\hat{\mathcal{F}}^2[f(x)] = f(-x)$. In other words,

   $$f(x) \xrightarrow{\hat{\mathcal{F}}} F(u) \xrightarrow{\hat{\mathcal{F}}} f(-x).$$

   So, calculating F gives us two transforms for the price of one!

   (b) Using properties of the Fourier transform only (no integrals!), prove the following:

      i. $\hat{\mathcal{F}}[\mathrm{sinc}] = \mathrm{rect}$.

      ii. $\hat{\mathcal{F}}[e^{2\pi iax}f(x)] = F(u-a)$.

      iii. $\hat{\mathcal{F}}[fg] = F * G$.

7. **Feynman's trick for Gaussians.** There is a neat trick for evaluating integrals that Feynman used to great effect; it is also called *differentiating under the integral sign* or *Leibniz's rule.* The rule is

   $$\frac{d}{d\alpha}\int_a^b f(x,\alpha)\,dx = \int_a^b \frac{\partial}{\partial\alpha} f(x,\alpha)\,dx,$$

   provided f is a smooth function and the limits a, b do not depend on α.

   (a) By differentiating both sides of (1) with respect to α, deduce that

   $$\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha^3}}.$$

   (b) *Differentiate n times to obtain

   $$\int_{-\infty}^{\infty} x^{2n} e^{-\alpha x^2}\,dx = \sqrt{\frac{\pi}{\alpha}}\,\frac{(2n-1)(2n-3)\cdots 3\cdot 1}{(2\alpha)^n}.$$

8. **The Uncertainty Principle.** You will have to wait until third year to see the full quantum-mechanical proof of Heiseinberg's uncertainty principle. However, there is a closely related result about Fourier transforms. As you learned in high school, a Gaussian of the form

   $$\frac{1}{\sqrt{2\pi\sigma^2}}e^{-x^2/2\sigma^2}$$

   has *standard deviation* σ. This measures the spread of the distribution. Let σ_x denote the spread of a spatial Gaussian wavepacket and σ_u the spread of its Fourier transform. Show that

   $$\sigma_x\sigma_u = \frac{1}{2\pi}.$$

9. **Fourier puzzles.***

   (a) Find all functions f with the property that $f * f = f$. Hint. Convolution theorem.

   (b) Using properties of the Fourier transform (or an explicit construction), find a *fixed point* of the Fourier transform, that is, a function f such that $\hat{\mathcal{F}}[f] = f$.

10. **The Klein-Gordon equation.** The 1D wave equation describes how disturbances propagate on a massless string. To describe a massive string, we need the Klein-Gordon operator:

$$\mathcal{L}_{\text{KG}} \equiv \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + 4\pi^2\lambda^2,$$

where λ is proportional to the density of the string. The *Klein-Gordon equation* is $\mathcal{L}_{\text{KG}} f = 0$. Recall that the Green's function ϕ for $\mathcal{L}_{\text{KG}}$ must satisfy

$$\mathcal{L}_{\text{KG}}\phi = -\delta(x)\delta(t). \tag{4}$$

By Fourier transforming (4) with respect to both time and space, show that the Greens' function $\hat{\phi}(k,\omega) \equiv \hat{\mathcal{F}}[\phi(x,t)]$ for the Klein-Gordon equation satisfies

$$\hat{\phi}(k,\omega) = -\frac{1}{4\pi^2}\frac{1}{k^2 - \omega^2 + \lambda^2}.$$

This tells us how a point disturbance spreads on a heavy string. Incidentally, it also describes the behaviour of elementary particles like the Higgs boson!

11. **The Schrödinger equation.*** The *Schrödinger equation* for a free particle

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} - i\hbar\frac{\partial\psi}{\partial t} = 0$$

is formally just a diffusion equation with imaginary coefficient $D = i\hbar/2m$. We showed in class that the fundamental solution of the Schrödinger equation was

$$\Phi_t(x) \equiv \Phi(x,t) = N(t)\exp\left(\frac{imx^2}{2\hbar t}\right),$$

where $N(t)$ is a time-dependent normalisation factor.

(a) At $t = 0$, suppose we have a Gaussian wavepacket:

$$\psi_0(x) \equiv \psi(x,0) = Ce^{-\kappa x^2}.$$

Use the fundamental solution and the method of Greens' functions to show that

$$\psi(x,t) = \int_{-\infty}^{\infty} \psi(\xi,0)\Phi(x-\xi,t)\,d\xi. \tag{5}$$

(b) We can rewrite (5) in the simple form $(\psi_0 * \Phi_t)$. Define $\gamma(t) \equiv -m/2\hbar t$, and assume that the identity in Problem 3(c) holds for imaginary α. Using the convolution theorem and Problem 3(c), prove

$$\psi(x,t) = CN(t)\sqrt{\frac{\pi}{\kappa + i\gamma(t)}}\exp\left(-\left[\frac{i\kappa\gamma(t)}{\kappa + i\gamma(t)}\right]x^2\right). \tag{6}$$

(c) The probability distribution for the particle is given by $|\psi|^2$. Ignoring the normalisation factors (which do not affect the shape) and focussing on the Gaussian part, show that

$$|\psi(x,t)|^2 \propto \exp\left(-\frac{x^2}{2\sigma^2(t)}\right), \quad \sigma^2(t) = \frac{\kappa\hbar^2 t^2}{m^2} + \frac{1}{4\kappa}.$$

Since $\sigma^2(t)$ directly measures the *spread* of the wavepacket, what is happening to the wavepacket as t increases?

PHYC20014 Physical Systems

Wave Theory and Fourier Analysis: Tutorial 3

Solutions

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1. **Basic Fourier transforms.**

   (a) No tricks here, we just plug and chug:

$$\begin{aligned}
\hat{\mathcal{F}}[\mathrm{rect}](u) &= \int_{-\infty}^{\infty} \mathrm{rect}(x) e^{-2\pi i u x}\, dx \\
&= \int_{-1/2}^{1/2} e^{-2\pi i u x}\, dx \\
&= -\frac{1}{2\pi i u}\left[e^{-2\pi i u x}\right]_{-1/2}^{1/2} dx \\
&= \frac{1}{2\pi i u}\left(e^{\pi i u} - e^{-\pi i u}\right) dx \\
&= \frac{\sin(\pi u)}{\pi u} \equiv \mathrm{sinc}(u).
\end{aligned}$$

   The function $\sin(\pi u)/\pi u$ is so ubiquitous that we give it a new name, the *sinc* function.

   (b) For this Fourier integral, we can use the fact that Λ is even, and the exponential splits into an even part $\cos(2\pi u x)$ and an odd part $\sin(2\pi u x)$:

$$\begin{aligned}
\hat{\mathcal{F}}[\Lambda](u) &= \int_{-\infty}^{\infty} \Lambda(x) e^{-2\pi i u x}\, dx \\
&= \int_{-1}^{1} (1 - |x|) e^{-2\pi i u x}\, dx \\
&= 2\int_{0}^{1} (1 - x)\cos(2\pi u x)\, dx \\
&= \frac{1 - \cos(2\pi u)}{2(\pi u)^2} \\
&= \frac{\sin^2(\pi u)}{(\pi u)^2} = \mathrm{sinc}^2(u).
\end{aligned}$$

   On the fourth line, we used integration by parts, and on the fifth, the double angle formula. An alternative approach is to use the convolution theorem (Problem 4(b)). First, calculate (visually as per Problem 4(a) or otherwise)

$$(\mathrm{rect} * \mathrm{rect})(x) = \int_{-\infty}^{\infty} \mathrm{rect}(\xi)\mathrm{rect}(x - \xi)\, d\xi = \Lambda(x).$$

   Then, by the convolution thorem

$$\hat{\mathcal{F}}[\Lambda](u) = \hat{\mathcal{F}}[\mathrm{rect} * \mathrm{rect}](u) = \hat{\mathcal{F}}[\mathrm{rect}](u)^2 = \mathrm{sinc}^2(u).$$

2. **Fourier transform properties.**

(a) Assume $a > 0$. Then, in the Fourier integral, make the change of variable $s = ax$:

$$\begin{aligned}\hat{\mathcal{F}}[f(ax)](u) &= \int_{-\infty}^{\infty} f(ax)e^{-2\pi iux}\,dx \\ &= \frac{1}{a}\int_{-\infty}^{\infty} f(s)e^{-2\pi i(u/a)s}\,ds = \frac{1}{a}F(u/a).\end{aligned}$$

For $a < 0$, the integration limits flip and we get the same result as above, but with a minus sign. Combining the two, we get the similiarity theorem:

$$\hat{\mathcal{F}}[f(ax)](u) = \frac{1}{|a|}F(u/a).$$

(b) Again, we shift variables $s = x - a$

$$\begin{aligned}\hat{\mathcal{F}}[f(x-a)](u) &= \int_{-\infty}^{\infty} f(x-a)e^{-2\pi iux}\,dx \\ &= \int_{-\infty}^{\infty} f(s)e^{-2\pi iu(s+a)}\,ds \\ &= e^{-2\pi ia}\int_{-\infty}^{\infty} f(s)e^{-2\pi iu(s+a}\,ds = e^{-2\pi ia}F(u).\end{aligned}$$

(c) Here, we need to assume that the function $f(x)$ vanishes as $x \to \pm\infty$. Then, integrating by parts,

$$\begin{aligned}\hat{\mathcal{F}}[f'(x)](u) &= \int_{-\infty}^{\infty} f'(x)e^{-2\pi iux}\,dx \\ &= \left[f(x)e^{-2\pi iux}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\frac{\partial}{\partial x}e^{-2\pi iux}\,dx \\ &= 2\pi iu\int_{-\infty}^{\infty} f(x)e^{-2\pi iux}\,dx = 2\pi iuF(u).\end{aligned}$$

3. **Gaussians.**

(a) As per usual, we are not too concerned with convergence. Then

$$\begin{aligned}I(\alpha)^2 &= \int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx\int_{-\infty}^{\infty} e^{-\alpha y^2}\,dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\alpha(x^2+y^2)}\,dx\,dy \\ &= \int_0^{\infty}\int_0^{2\pi} e^{-\alpha r^2}(r\,d\phi\,dr) \\ &= 2\pi\int_0^{\infty} e^{-\alpha r^2}r\,dr) = -\frac{\pi}{\alpha}\left[e^{-\alpha r^2}\right]_0^{\infty} = \frac{\pi}{\alpha}.\end{aligned}$$

Hence,

$$I(\alpha) = \sqrt{\frac{\pi}{\alpha}}.$$

(b) First, note that

$$-\alpha x^2 + \beta x = -\alpha \left(x - \frac{\beta}{2\alpha} \right)^2 + \frac{\beta^2}{4\alpha}.$$

Now we just shift variables $s = x - \beta/2\alpha$ and use part (a):

$$\int_{-\infty}^{\infty} e^{-\alpha x^2 + \beta x}\, dx = \exp\left(\frac{\beta^2}{4\alpha}\right) \int_{-\infty}^{\infty} e^{-\alpha s^2}\, ds$$
$$= \sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta^2}{4\alpha}\right).$$

To evaluate the last integral, we change $\alpha \to \alpha - i\beta$ and $\beta \to -2\pi i u$, and use (2):

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} e^{i\beta x^2} e^{-2\pi i u x}\, dx = \sqrt{\frac{\pi}{\alpha - i\beta}} \exp\left(-\frac{\pi^2 u^2}{\alpha - i\beta}\right).$$

(c) We take the Fourier transform, and set $\beta = -2\pi i u$ in the result in (b):

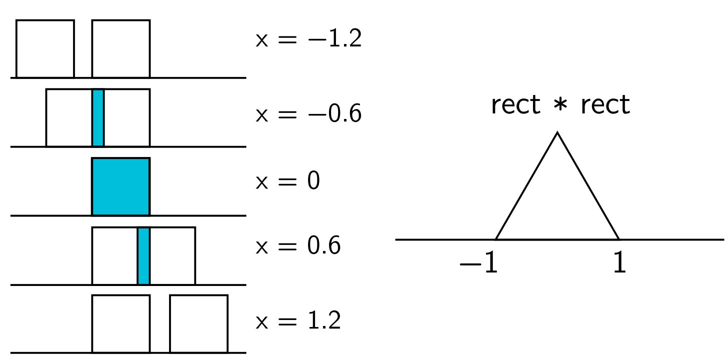
$$\hat{\mathcal{F}}[e^{-\alpha x^2}](u) = \int_{-\infty}^{\infty} e^{-\alpha x^2} e^{-2\pi i u x}\, dx = \sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta^2}{4\alpha}\right) = \sqrt{\frac{\pi}{\alpha}} \exp\left(-\frac{\pi^2 u^2}{\alpha}\right).$$

4. **Convolutions.**

(a) We sneakily factorise $e^{-2\pi i u x}$ as suggested, and make the change of variables $s = x - \xi$:

$$\begin{aligned}
\hat{\mathcal{F}}[f * g](u) &= \int_{-\infty}^{\infty} (f * g)(x) e^{-2\pi i u x}\, dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\xi) f(x - \xi)\, d\xi e^{-2\pi i u x}\, dx \\
&= \int_{-\infty}^{\infty} g(\xi) e^{-2\pi i u \xi} \int_{-\infty}^{\infty} f(x - \xi) e^{-2\pi i u (x - \xi)}\, d\xi, dx \\
&= \int_{-\infty}^{\infty} g(\xi) e^{-2\pi i u \xi}\, d\xi \int_{-\infty}^{\infty} f(s) e^{-2\pi i u s}\, ds = \hat{\mathcal{F}}[g](u) \hat{\mathcal{F}}[f](u).
\end{aligned}$$

(b) To evaluate the convolution $(f * g)(x)$ visually, we first take a copy of the function f. We then take a copy of g, flip it with respect to the x axis, and shift it to the right by x. Finally, we multiply these two functions (regular f and shifted, flipped g) and compute the integral over $\mathbb{R}$. We illustrate this using $f = g = \text{rect}$ below:



The convolution is clearly piecewise linear, and consideration of the diagram shows that

$$\operatorname{rect} * \operatorname{rect}(x) = \begin{cases} 0 & x < -1 \\ 1 + x & -1 \leq x \leq 0 \\ 1 - x & 0 < x < 1 \\ 0 & x \geq 1 \end{cases}.$$

But this is just a long-winded description of $\Lambda(x)$, which we encountered in Problem 1.

5. **Heaviside step and Dirac delta.**

(a) For a test function $f(x)$ which is differentiable and vanishes at $x \to \pm\infty$, integration by parts and the fundamental theorem of calculus yield

$$\begin{aligned} \int_{-\infty}^{\infty} \Theta'(x) f(x)\, dx &= -\int_{-\infty}^{\infty} \Theta(x) f'(x)\, dx \\ &= -\int_{0}^{\infty} f'(x)\, dx = -[f(x)]_0^\infty = f(0). \end{aligned}$$

This is precisely the sifting property of $\delta(x)$, so $\Theta'(x) = \delta(x)$.

(b) Using $\hat{\mathcal{F}}[\delta] = 1$ and the derivative theorem, part (a) implies

$$\hat{\mathcal{F}}[\Theta] = \frac{1}{2\pi i u} \hat{\mathcal{F}}[\Theta'] = \frac{1}{2\pi i u} \hat{\mathcal{F}}[\delta] = \frac{1}{2\pi i u}.$$

Incidentally, combining this with the shift theorem, we get another way to Fourier transform of rect, since

$$\operatorname{rect}(x) = \frac{1}{2}\left[\Theta\left(x - \frac{1}{2}\right) + \Theta\left(\frac{1}{2} - x\right)\right].$$

I leave the details to the interested reader.

6. **More fun with Fourier transforms.**

(a) A simple way to prove this is to use the Fourier transform representation of the delta function:

$$\delta(x) = \int_{-\infty}^{\infty} e^{2\pi u x}\, du.$$

So, it follows that

$$\begin{aligned} \hat{\mathcal{F}}^2[f](x) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dy\, du\, f(y) e^{-2\pi i u y} e^{-2\pi i u x} \\ &= \int_{-\infty}^{\infty} dy\, f(y) \left(\int_{-\infty}^{\infty} du\, e^{-2\pi i u (x+y)} \right) \\ &= \int_{-\infty}^{\infty} dy\, f(y) \delta(-(x+y)) = \int_{-\infty}^{\infty} dy\, f(y) \delta(x+y) = f(-x). \end{aligned}$$

On the last line, we used the fact that δ is even.

(b) i. Since rect is even, this follows immediately from Problem 1(a) and 6(a).

ii. Take the Fourier transform of the RHS, and use Problem 2(b) (the shift theorem) and 6(a):

$$\hat{\mathcal{F}}[F(u-a)](x) = \hat{\mathcal{F}}[F](x)e^{-2\pi ixa} = f(-x)e^{-2\pi ixa}.$$

But this is just the Fourier transform of the LHS. Hence, the two sides must be equal.

iii. Here, we Fourier transform both sides and check they are equal, using Problem 4(b) (the convolution theorem) and 6(a) again. For convenience, let $f_-(x) \equiv f(-x)$. We then have

$$\hat{\mathcal{F}}^2[fg] = f_- g_- = \hat{\mathcal{F}}[F]\hat{\mathcal{F}}[G] = \hat{\mathcal{F}}[F * G],$$

as required.

7. **The Uncertainty Principle.** We use the results of Problem 3(b) with

$$\alpha = \frac{1}{2\sigma_x^2}, \quad \frac{\pi^2}{\alpha} = \frac{1}{2\sigma_u^2}.$$

Combining the two,

$$\sigma_x \sigma_u = \frac{1}{2\pi}.$$

It turns out (though we will not prove it here) that the Gaussian transform pair minimises the product $\sigma_x \sigma_u$, so in general

$$\sigma_x \sigma_u \geq \frac{1}{2\pi}.$$

This is clearly analogous to the uncertainty principle. In signal processing, this is called the *time-bandwidth relation.*

8. **Feynman's trick for Gaussians.** We prove part (b) only; (a) is a special case. Differentiate both sides of (1) with respect to α, n times. Using Feynman's trick, the LHS is

$$\frac{d^n}{d\alpha^n}\int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx = \int_{-\infty}^{\infty} \frac{\partial^n}{\partial \alpha^n} e^{-\alpha x^2}\,dx = (-1)^n \int_{-\infty}^{\infty} x^{2n} e^{-\alpha x^2}\,dx.$$

The RHS is

$$\frac{d^n}{d\alpha^n}\sqrt{\frac{\pi}{\alpha}} = \sqrt{\pi}\frac{d^n}{d\alpha^n}\alpha^{-1/2} = (-1)^n \alpha^{-n-1/2}\sqrt{\pi}\frac{(2n-1)(2n-3)\cdots 3\cdot 1}{2^n}.$$

Equating the two and dividing by $(-1)^n$ gives the desired result,

$$\int_{-\infty}^{\infty} x^{2n} e^{-\alpha x^2}\,dx = \sqrt{\frac{\pi}{\alpha}}\frac{(2n-1)(2n-3)\cdots 3\cdot 1}{(2\alpha)^n}.$$

9. **Fourier puzzles.***

(a) From the convolution theorem, if $f * f = f$, then $F^2 = F$, where $F \equiv \hat{\mathcal{F}}[f]$. It follows that, at each point u, $F(u)$ must be 1 or 0. For instance, since $\hat{\mathcal{F}}[\delta] = 1$, it follows that $\delta * \delta = \delta$; this is also easy to see directly. Less trivially, since $\hat{\mathcal{F}}[\text{sinc}] = \text{rect}$, and rect is always 1 or 0, we have the nonobvious result $\text{sinc} * \text{sinc} = \text{sinc}$.

(b) There are lots of ways to do this. One is to use the result of Problem 3(c) to check that the Gaussian

$$f(x) = e^{-\pi x^2}$$

is its own Fourier transform. A more general construction is as follows: take any *even* function f and its Fourier transform F, and add them together. Then

$$f(x) + F(x) \xrightarrow{\hat{\mathcal{F}}} F(u) + f(-u) = F(u) + f(u)$$

using part (a) and the fact that f is even. So, there is no shortage of fixed points!

10. **The Klein-Gordon equation.** We will become intimately familiar with the 2D Fourier transform in the context of optics. Mathematically, it is just two separate Fourier transforms:

$$\hat{\phi}(k,\omega) \equiv \hat{\mathcal{F}}[\phi(x,t)](k,\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(x,t)e^{-2\pi ikx}e^{-2\pi i\omega x}\,dx\,dt.$$

The variables *independently* satisfy the relations in Problem 2, in particular the derivative theorem:

$$\begin{aligned}\hat{\mathcal{F}}[\mathcal{L}_{\mathrm{KG}}\phi](k,\omega) &= \hat{\mathcal{F}}\left[\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + 4\pi^2\lambda^2\right)\phi\right](k,\omega) \\ &= 4\pi^2\left(k^2 - \omega^2 + \lambda^2\right)\hat{\phi}(k,\omega).\end{aligned}$$

Suppose that ϕ is a fundamental solution as per (4). Since $\hat{\mathcal{F}}[\delta(x)\delta(t)] = \hat{\mathcal{F}}[\delta(x)]\hat{\mathcal{F}}[\delta(t)] = 1$, we have

$$\begin{aligned}-1 &= 4\pi^2\left(k^2 - \omega^2 + \lambda^2\right)\hat{\phi}(k,\omega) \\ \implies \quad \hat{\phi}(k,\omega) &= -\frac{1}{4\pi^2}\frac{1}{k^2 - \omega^2 + \lambda^2}.\end{aligned}$$

11. **The Schrödinger equation.***

(a) For a linear PDE, we can write the general solution as an integral over (shifted) fundamental solutions; this is the continuous analogue of what we do in ODEs. We then pick the coefficients of the integral to match our initial conditions. In this case, our initial condition is $\psi(x,0)$. At each x, we place a delta function $\delta(x-\xi)$, and evolve them independently using the fundamental solution:

$$\psi_0(x) = \int_{-\infty}^{\infty} \psi_0(\xi)\delta(x-\xi)\,d\xi \quad \implies \quad \psi(x,t) = \int_{-\infty}^{\infty} \psi_0(\xi)\Phi(x-\xi,t)\,d\xi.$$

(b) Let $\hat{\mathcal{F}}$ indicate the Fourier transform with respect to x only. Define $\gamma(t) \equiv -m/2\hbar t$. By the convolution theorem, Problem 3(c), and part (a),

$$\begin{aligned}\hat{\mathcal{F}}[\psi(x,t)](u) = \hat{\mathcal{F}}[\psi_0 * \Phi_t](u) &= \hat{\mathcal{F}}[\psi_0](u)\hat{\mathcal{F}}[\Phi_t](u) \\ &= CN(t)\sqrt{\frac{\pi}{\kappa}}\sqrt{\frac{\pi}{i\gamma}}\exp\left(-\frac{\pi^2u^2}{\kappa}\right)\exp\left(-\frac{\pi^2u^2}{i\gamma}\right) \\ &= CN(t)\frac{\pi}{\sqrt{\kappa i\gamma}}\exp\left(-\left[\frac{\kappa+i\gamma}{i\kappa\gamma}\right]\pi^2u^2\right).\end{aligned}$$

Again using Problem 3(c), we can invert this by inspection:

$$\psi(x,t) = CN(t)\sqrt{\frac{\pi}{\kappa + i\gamma}}\exp\left(-\left[\frac{i\kappa\gamma}{\kappa + i\gamma}\right]x^2\right).$$

(c) For $z = e^{x+iy}$, $|z|^2 = e^{2x}$. Thus, we need to find the real part of the exponential coefficient in (6):

$$\mathrm{Re}\left[\frac{i\kappa\gamma}{\kappa + i\gamma}\right] = \mathrm{Re}\left[\frac{i\kappa\gamma(\kappa - i\gamma)}{\kappa^2 + \gamma^2}\right] = \frac{\kappa\gamma^2(t)}{\kappa^2 + \gamma^2(t)}.$$

Finally, we can determine the spread:

$$\sigma^2(t) = \frac{\kappa^2 + \gamma^2(t)}{4\kappa\gamma^2(t)} = \frac{\kappa^2 + (m^2/4\hbar^2 t^2)}{4\kappa(m^2/4\hbar^2 t^2)} = \frac{\kappa\hbar^2 t^2}{m^2} + \frac{1}{4\kappa}.$$

We see that, over time, the wavepacket spreads out.

PHYC20014 Physical Systems

Wave Theory and Fourier Analysis: Tutorial 4

Tutorial problems

1. **Cross-correlations.** The *cross-correlation* of two functions f and g is defined by

$$(f \otimes g)(x) \equiv \int_{-\infty}^{\infty} f^*(\xi) g(\xi + x)\, d\xi = \int_{-\infty}^{\infty} f^*(\xi - x) g(\xi)\, d\xi.$$

It is a simple way to measure the similarity of two signals f and g, peaking at the offsets x for which the functions are most similar.

   (a) Express the cross-correlation as a convolution.

   (b) Define $f_-(x) \equiv f(-x)$. Show that

$$\hat{\mathcal{F}}[f_-^*] = F^*.$$

   (c) From (a) and (b), prove that the cross-correlation satisfies

$$\hat{\mathcal{F}}[f \otimes g] = F^* \cdot G.$$

2. **The DFT and Fourier series.** Consider a sequence of N complex numbers, $\mathbf{x} = (x_0, x_1, \ldots, x_{N-1})$. Typically, these numbers are obtained by *sampling* a continuous function, $x_n = f(t_n)$. The *discrete Fourier transform* (DFT) is another sequence $\mathbf{X} = (X_0, X_1, \ldots, X_{N-1})$, defined by

$$X_k \equiv \sum_{n=0}^{N-1} x_n e^{-i\omega kn}, \quad \omega \equiv \frac{2\pi}{N}.$$

   (a) Verify that the inverse transform is

$$x_k = \frac{1}{N} \sum_{n=0}^{N-1} X_n e^{i\omega kn}.$$

   (b) *Generate a sequence x_n by sampling a function f of period T at intervals T/N:

$$x_n \equiv f\left(\theta_n\right), \quad \theta_n \equiv \frac{nT}{N}.$$

   Argue that in the limit $N \to \infty$, the DFT is related to the exponential Fourier series for f as follows:

$$X_k \longrightarrow Tc_k = \int_0^T f(\theta) e^{-i(2\pi/T)k\theta}\, d\theta.$$

3. **Autocorrelations.** The *autocorrelation* of a function f is the cross-correlation with itself, $f \otimes f$. Autocorrelations let us look for repeating patterns in a signal over time. Hence, they can help us filter out noise, which is *not* correlated with itself over time.

(a) Specialise Problem 1(b) to show that

$$\hat{\mathcal{F}}[f \otimes f] = |F|^2.$$

This is the *Wiener-Khinchin theorem*. It states that the *spectral density* $|F|^2$ is the Fourier transform of the autocorrelation function $f \otimes f$.

(b) Let $f(x) = \sin(2\pi\nu x)$. Show that $F(u) = (i/2)[\delta(u+\nu) - \delta(u-\nu)]$.

(c) Using the results of Problem 4(a), show that

$$(f \otimes f)(x) = \frac{1}{2}\cos(2\pi\nu x).$$

HINT. You may assume that $\delta(x)^2 = \delta(x)$ and $\delta(x)\delta(x') = 0$ for $x \neq x'$.

(d) Interpret this result in terms of repeating patterns in the signal $f(x) = \sin(2\pi\nu x)$.

4. **Lifetime broadening.** An atom is excited by collision with a photon of energy E at $t = 0$. The lifetime of the excited state is τ, so the probability $p(t)$ of being excited state at time t is

$$p(t) = \begin{cases} e^{-t/\tau} & t \geq 0 \\ 0 & t < 0. \end{cases}$$

Let $P \equiv \hat{\mathcal{F}}[p]$. The uncertainty in the time of emission (due to the finite lifetime of the excited state) leads to uncertainty in the energy of emitted photons, a phenomenon called *lifetime broadening*. The *spectral line shape* $I(u) \equiv |P(u)|^2$ measures the range of emitted frequencies.

(a) Show that

$$I(u) = \frac{1}{4\pi^2}\frac{1}{u^2+\gamma^2}, \quad \gamma \equiv \frac{1}{2\pi\tau}.$$

Here, γ is the *decay rate* and $I(u)$ is called the *Lorentz profile.*

(b) Show that the maximum value of $I(u)$ is τ^2, and the width of the graph (in u) at half the maximum value is 2γ. Thus, 2γ is sometimes called the *full width at half-maximum.*

5. **Bravais lattices and reciprocals.** A *Bravais lattice* $\mathbf{R}$ is a periodic array of points. In 2D, the simplest example is the set of integer linear combinations of basis vectors $\mathbf{a}_1 \equiv (x_1, y_1)$ and $\mathbf{a}_2 \equiv (x_2, y_2)$:

$$\mathbf{R} \equiv \{n_1\mathbf{a}_1 + n_2\mathbf{a}_2 : n_1, n_2 \in \mathbb{Z}\}.$$

The *reciprocal lattice* $\mathbf{R}^\perp$ is the set of vectors $\mathbf{k}$ such that, for any $\mathbf{r} \in \mathbf{R}$,

$$e^{2\pi i\mathbf{k}\cdot\mathbf{r}} = 1. \tag{1}$$

(a) Let's consider the 2D Bravais lattice $\mathbf{R}$ given above. Define the vectors

$$\mathbf{b}_1 \equiv \frac{(-y_2, x_2)}{x_2y_1 - x_1y_2}, \quad \mathbf{b}_2 \equiv \frac{(-y_1, x_1)}{x_1y_2 - x_2y_1}.$$

Check that $\mathbf{a}_i \cdot \mathbf{b}_j = \delta_{ij}$.

(b) Conclude that $\mathbf{R}^\perp$ is a Bravais lattice generated by $\mathbf{b}_1$ and $\mathbf{b}_2$.

(c) *Consider a function f defined on the lattice $\mathbf{R}$. We can extend the definition to all of space using Dirac deltas:

$$f(\mathbf{x}) = \sum_{\mathbf{r}\in\mathbf{R}} c_{\mathbf{r}}\delta(\mathbf{x}-\mathbf{r}).$$

Show that the Fourier transform $F = \hat{\mathcal{F}}[f]$ is periodic on $\mathbf{R}^{\perp}$. For this reason, the reciprocal lattice $\mathbf{R}^{\perp}$ is often called the Fourier transform of $\mathbf{R}$. Hint. Shift theorem.

Extra problems

6. **Simple signal processing.** In digital signal processing, we Fourier transform a signal f, apply a *filter* K, then invert:

$$f \xrightarrow{\hat{\mathcal{F}}} F \xrightarrow{K} KF \xrightarrow{\hat{\mathcal{F}}^{-1}} f_K \equiv \hat{\mathcal{F}}^{-1}[KF].$$

Sketch filters which perform the following tasks for *sound* waves. For simplicity, consider only positive frequencies:

(a) Pump up the bass ($\omega < \omega_B$) and decrease the treble ($\omega > \omega_T$).

(b) Filter out the high-pitched whine of buzzsaw nearby (frequency ω_S, spread $\Delta\omega$).

(c) Autotune a voice to a base note ω_0 and harmonics thereof.

7. **Harmonic functions.** As discussed in lectures, there is a deep connection between harmonic and analytic functions: if $f(z) = u(z)+iv(z)$ is analytic, then $u(x, y)$ and $v(x, y)$ are harmonic, with orthogonal contours (level sets).

(a) Recall that, in polar coordinates, we can write a complex number as $z = re^{i\theta}$. Assuming that the complex logarithm is analytic, use the connection between analytic and harmonic functions to deduce that

$$u(x, y) = \log r(x, y), \quad v(x, y) = \theta(x, y)$$

form a conjugate harmonic pair.[1] Draw the level sets and confirm they are perpendicular.

(b) Consider a coaxial cable with outer radius R_1 and inner radius R_2; these are held at constant potentials V_1 and V_2 respectively. Inside the cable there are no charges, so the potential is harmonic:

$$\nabla^2 V = 0.$$

Confining your attention to a 2D cross-section of the cable, solve the boundary value problem. Hint. Use one of the functions in (a).

(c) *Without doing it*, explain how would you solve Laplace's equation for V on an infinite wedge $\theta_1 < \theta < \theta_2$ with specified values V_1, V_2 on the boundary rays.

8. **Whittaker-Shannon sampling theorem.*** A function f is *bandlimited of width* L if its Fourier transform F is only nonzero on a finite interval of length $\leq L$. It turns out that we can reconstruct f in its entirety from a discrete (though infinite) set of samples. This is called the *Whittaker-Shannon sampling theorem*, and this problem steps you through the proof.

[1] There are some subtleties to do with $\theta(x, y)$ and analyticity we are sweeping under the rug; they do not affect the main results of the problem.

(a) Recall the Dirac comb Ш_T from Tutorial 1. Show that

$$\hat{\mathcal{F}}[\text{Ш}_T] = \frac{1}{T}\text{Ш}_{1/T}.$$

(b) We can sample values of f, at intervals T, simply by multiplying by the Dirac comb:

$$f_T(x) = \text{Ш}_T(x)f(x).$$

Using the convolution theorem, prove that

$$\hat{\mathcal{F}}[f_T] = \frac{1}{T}\sum_{k=-\infty}^{\infty} F\left(u - \frac{k}{T}\right).$$

(c) So far, we have assumed nothing about f. If f is bandlimited, argue that sampling at intervals $T \leq L^{-1}$ separates the individual copies of F in the Fourier transform $\hat{\mathcal{F}}[f_T]$. From F, we can reconstruct the entire function f using the inverse Fourier transform!

9. **Diagonalisation and eigenbases.*** We can view the Fourier transform in the same way we view an expansion in orthogonal polynomials: expansion in a well-chosen basis for a (very large) vector space V. In this case, the basis elements are exponentials are rather than polynomials,

$$f_\lambda(u) = e^{2\pi i\lambda u}.$$

Usually, we choose a basis consisting of eigenvectors of some linear operator $\mathcal{L} : V \to V$. This is the *eigenbasis* corresponding to $\mathcal{L}$.

(a) Explain why $\mathcal{L}$ is said to be diagonal in its eigenbasis.

(b) Find a simple operator $\mathcal{L}$ for which the exponentials $\{f_\lambda : \lambda \in \mathbb{R}\}$ form an eigenbasis.

10. **3D Bravais lattices.*** Consider a Bravais lattice in 3D given by

$$\mathbf{R} \equiv \{n_1\mathbf{a}_1 + n_3\mathbf{a}_3 + n_3\mathbf{a}_3 : n_1, n_2, n_3 \in \mathbb{Z}\}.$$

(a) Show that the reciprocal lattice $\mathbf{R}^\perp$ has basis vectors

$$\mathbf{b}_1 = \frac{\mathbf{a}_2 \times \mathbf{a}_3}{\mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{a}_3)}, \quad \mathbf{b}_2 = \frac{\mathbf{a}_3 \times \mathbf{a}_1}{\mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{a}_3)}, \quad \mathbf{b}_3 = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{\mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{a}_3)}.$$

(b) A *simple cubic* Bravais lattice of side a has lattice vectors $\mathbf{a}_i = a\hat{\mathbf{x}}_i$ describing the sides of a cube. Show that the reciprocal lattice is a simple cubic lattice with side length $1/a$.

(c) A *body-centred cubic* (bcc) lattice of side a has basis vectors

$$\mathbf{a}_i = \frac{a}{2}(\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}} - 2\hat{\mathbf{x}}_i).$$

A *face-centred cubic* (fcc) lattice of side a has basis vectors

$$\mathbf{a}_i = \frac{a}{2}(\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}} - \hat{\mathbf{x}}_i).$$

Show that the reciprocal of a bcc lattice of side a is an fcc lattice of side $2/a$.

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Wave Theory and Fourier Analysis: Tutorial 4

Solutions

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1. **Cross-correlations.**

   (a) Comparing the definitions,

$$(f * g)(x) \equiv \int_{-\infty}^{\infty} f(\xi - x) g(\xi)\, d\xi$$
$$(f \otimes g)(x) \equiv \int_{-\infty}^{\infty} f^*(x - \xi) g(\xi)\, d\xi,$$

   we see that $f \otimes g = f_-^* * g$.

   (b) Using basic properties of the complex conjugate,

$$\begin{aligned} F^*(u) &= \left[\int_{-\infty^{\infty}} e^{-2\pi i u x} f(x)\, dx \right]^* \\ &= \int_{-\infty}^{\infty} \left(e^{-2\pi i u x} \right)^* f^*(x)\, dx \\ &= \int_{-\infty}^{\infty} e^{2\pi i u x} f^*(x)\, dx \\ &= -\int_{\infty}^{-\infty} e^{-2\pi i u s} f^*(-s)\, ds \\ &= \int_{-\infty}^{\infty} e^{-2\pi i u s} f_-^*(s)\, ds = \hat{\mathcal{F}}[f_-^*](u). \end{aligned}$$

   On the fourth line, we made the change of variable $s = -x$.

   (c) From parts (a), (b) and the convolution theorem,

$$\hat{\mathcal{F}}[f \otimes g] = \hat{\mathcal{F}}[f_-^* * g] = \hat{\mathcal{F}}[f_-^*]\hat{\mathcal{F}}[g] = \hat{\mathcal{F}}[f]^* \hat{\mathcal{F}}[g] = F^* G.$$

2. **The DFT and Fourier series.**

   (a) We use the geometric series:

$$\begin{aligned} \frac{1}{N} \sum_{n=0}^{N-1} X_n e^{i\omega k n} &= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{j=0}^{N-1} x_j e^{i\omega(k-j)n} \\ &= \frac{1}{N} \sum_{j=0}^{N-1} x_j \sum_{n=0}^{N-1} e^{i\omega(k-j)n} \\ &= x_k + \frac{1}{N} \sum_{j=0, j\neq k}^{N-1} x_j \frac{1 - e^{i\omega(k-j)N}}{1 - e^{i\omega(k-j)}} = x_k. \end{aligned}$$

   On the last line, we used the fact that $e^{i\omega(k-j)N} = e^{2\pi i(k-j)} = 1$.

(b) Let's substitute the definitions of x_n, θ_n into the definition of X_k:

$$X_k = \sum_{n=0}^{N-1} f(\theta_n)\, e^{-i(2\pi/T)k\theta_n}.$$

Here, θ_n ranges from 0 to $T(N-1)/N$, and the step size is $\Delta\theta = T/N$. As $N \to \infty$, the sum becomes an integral with $\Delta\theta \to d\theta$. Let's check the normalisations agree:

$$\sum_{n=0}^{N-1} \Delta\theta = \int_0^T d\theta = T.$$

Putting it all together,

$$X_k \quad \longrightarrow \quad \int_0^T f(\theta)\, e^{-i(2\pi/T)k\theta}\, d\theta = Tc_k.$$

3. **Autocorrelations.**

(a) Since $f = g$, $\hat{\mathcal{F}}[f \otimes f] = F^* F = |F|^2$.

(b) Here we can use $\hat{\mathcal{F}}[1] = \delta(u)$ and Problem 6(b)(iii) from Tutorial 3:

$$\begin{aligned}\hat{\mathcal{F}}[\sin(2\pi\nu x)](u) &= \frac{1}{2i}\left\{\hat{\mathcal{F}}[e^{2\pi i\nu x}](u) - \hat{\mathcal{F}}[e^{-2\pi i\nu x}](u)\right\}\\ &= \frac{i}{2}\left\{\delta(u+\nu) - \delta(u-\nu)\right\}.\end{aligned}$$

(c) From the Wiener-Khinchin theorem, the Fourier transform of the autocorrelation is

$$\begin{aligned}\hat{\mathcal{F}}[f \otimes f] = |F|^2 &= \frac{1}{4}\left|\delta(u+\nu)^2 - 2\delta(u+\nu)\delta(u-\nu) + \delta(u-\nu)^2\right|\\ &= \frac{1}{4}\left\{\delta(u+\nu) + \delta(u-\nu)\right\},\end{aligned}$$

where we used the hints for the last equality. This is just the Fourier transform of $\cos(2\pi\nu x)/2$, as you can directly verify. Hence,

$$f \otimes f = \frac{1}{2}\cos(2\pi\nu x).$$

(d) Let $T \equiv \nu^{-1}$ denote the period of the original sine signal. The result in (c) states that the autocorrelation (repetition in the signal) peaks for offsets $0, \pm T, \pm 2T, \ldots$. In other words, it is periodic with period T! This hopefully makes sense: a sine signal will match itself exactly if we shift it backward or forward by a multiple of the period.

You may have observed that the assumption $\delta(x)^2 = \delta(x)$ is somewhat dodgy; in fact, using the sifting property, it is easy to see this is *false*, and $\delta(x)^2$ is not even a well-defined distribution. So, there are some infinities (corresponding to infinite integrals) we are sweeping under the rug! The main point, however, is the periodicity of the autocorrelation, which remains true even when we include infinities.

4. **Lifetime broadening.**

   (a) First, we calculate $P = \hat{\mathcal{F}}[p]$:

$$\begin{aligned} P(u) &= \int_{-\infty}^{\infty} p(t) e^{-2\pi i u t}\, dt \\ &= \int_{0}^{\infty} e^{-(2\pi i u + \tau^{-1})t}\, dt \\ &= -\frac{1}{2\pi i u + \tau^{-1}} \left[e^{-(2\pi i u+\tau^{-1})t} \right]_0^\infty \\ &= \frac{1}{2\pi i u + \tau^{-1}}. \end{aligned}$$

   Hence,

$$I(u) = |P(u)|^2 = \frac{1}{4\pi^2 u^2 + \tau^{-2}} = \frac{1}{4\pi^2} \frac{1}{u^2+\gamma^2}.$$

   (b) The maximum occurs at $u = 0$, with $I(0) = 1/(2\pi\gamma)^2 = \tau^2$. Solving for the half-maximum,

$$I(u) = \frac{1}{4\pi^2}\frac{1}{u^2+\gamma^2} = \frac{\tau^2}{2} \quad \Longrightarrow \quad u = \pm\gamma.$$

   Thus, the full width at half-maximum is 2γ.

5. **Bravais lattices and reciprocals.**

   (a) Let's calculate for $\mathbf{a}_i$:

$$\begin{aligned} \mathbf{a}_1 \cdot \mathbf{b}_1 &= (x_1, y_1) \cdot \frac{(-y_2, x_2)}{x_2 y_1 - x_1 y_2} = 1 \\ \mathbf{a}_1 \cdot \mathbf{b}_2 &= (x_1, y_1) \cdot \frac{(-y_1, x_1)}{x_1 y_2 - x_2 y_1} = 0. \end{aligned}$$

   Similarly, $\mathbf{a}_2 \cdot \mathbf{b}_1 = 0$, $\mathbf{a}_2 \cdot \mathbf{b}_2 = 1$, and $\mathbf{a}_i \cdot \mathbf{b}_j = \delta_{ij}$ as claimed.

   (b) The Bravais lattice $\mathbf{R}^\perp$ is a Bravais lattice generated by $\mathbf{b}_1$ and $\mathbf{b}_2$:

$$\mathbf{R}^\perp = \{m_1 \mathbf{b}_1 + m_2 \mathbf{b}_2 : m_1, m_2 \in \mathbb{Z}\}.$$

   Now we just check using part (a) that $\mathbf{R}$ and $\mathbf{R}^\perp$ satisfy (1). For arbitrary elements $\mathbf{r} = n_1\mathbf{a}_1 + n_2\mathbf{a}_2 \in \mathbf{R}$ and $\mathbf{k} = m_1\mathbf{b}_1 + m_2\mathbf{b}_2 \in \mathbf{R}^\perp$,

$$\begin{aligned} \exp\left[2\pi i \mathbf{r}\cdot\mathbf{k}\right] &= \exp\left[2\pi i (n_1\mathbf{a}_1 + n_2\mathbf{a}_2)\cdot(m_1\mathbf{b}_1 + m_2\mathbf{b}_2)\right] \\ &= \exp(2\pi i m_1 n_1)\exp(2\pi i m_2 n_2) = 1. \end{aligned}$$

   (c) Using linearity, $\hat{\mathcal{F}}[\delta] = 1$ and the shift theorem,

$$F(\mathbf{u}) = \sum_{\mathbf{r}\in\mathbf{R}} c_{\mathbf{r}} \hat{\mathcal{F}}[\delta(\mathbf{x}-\mathbf{r})] = \sum_{\mathbf{r}\in\mathbf{R}} e^{2\pi i \mathbf{u}\cdot\mathbf{r}} c_{\mathbf{r}}.$$

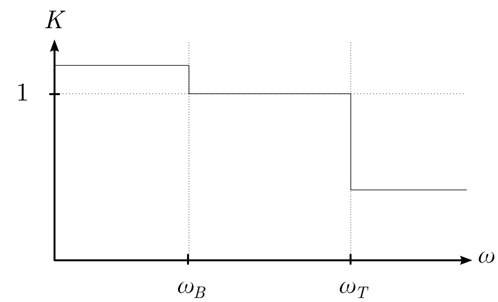
In fact, we have used a version of the shift theorem valid in any number of dimensions, but if you are more comfortable, feel free to specialise to 1D. For any element $\mathbf{k} \in \mathbf{R}^{\perp}$, we then have

$$F(\mathbf{u}+\mathbf{k}) = \sum_{\mathbf{r}\in\mathbf{R}} e^{2\pi i(\mathbf{u}+\mathbf{k})\cdot\mathbf{r}} c_{\mathbf{r}} = \sum_{\mathbf{r}\in\mathbf{R}} e^{2\pi i\mathbf{u}\cdot\mathbf{r}} e^{2\pi i\mathbf{k}\cdot\mathbf{r}} c_{\mathbf{r}} = \sum_{\mathbf{r}\in\mathbf{R}} e^{2\pi i\mathbf{u}\cdot\mathbf{r}} c_{\mathbf{r}} = F(\mathbf{u})$$

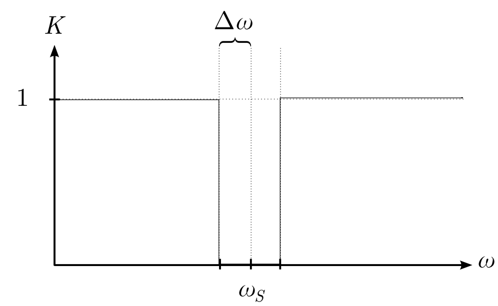
where we used (1). As required, F is periodic on $\mathbf{R}^{\perp}$.

6. **Simple signal processing.**

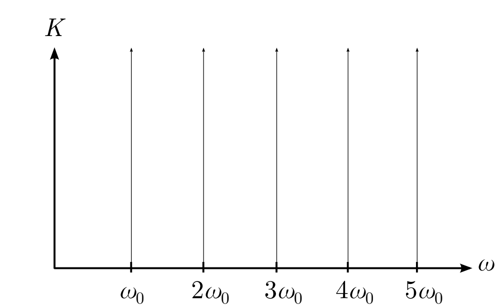
(a) We want to increase the amplitude of frequencies $\omega < \omega_B$ and decrease the amplitude for $\omega > \omega_T$. So, we want something like:



(b) Here, we simply want to throw away all the spectral information in a band of width $\Delta\omega$ centred at ω_S, so we would have:



(c) This is the opposite of (b): we only want to keep certain frequencies. So, our filter will be a Dirac comb, straining out the frequencies $n\omega_0$, $n = 1, 2, 3, \ldots$:

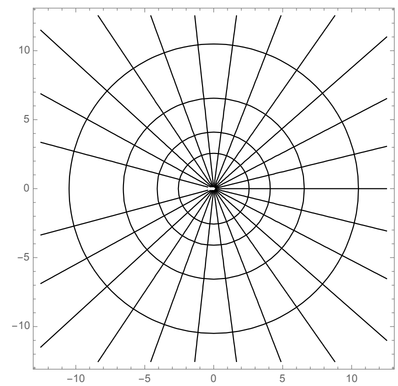


7. **Harmonic functions.**

(a) We assume that the complex logarithm is analytic without worrying too much.[2] Then

$$\log z = \log(re^{i\theta}) = \log r(x, y) + i\theta(x, y).$$

Since log is analytic, from the result in lectures, $\log\sqrt{x^2 + y^2}$ and $\theta(x, y)$ are harmonic, and have orthogonal contours. Contours $\log r = \text{const}$ are circles around the origin, while $\theta = \text{const}$ are rays from the origin, which are indeed perpenducular:



The more skeptical among you may be worried about invoking the magic of complex analysis without checking anything. Explicitly, $\log r(x, y) = \log\sqrt{x^2 + z^2}$ and, at least in the first quadrant, $\theta(x, y) = \arctan(y/x)$. You are invited to verify that these are harmonic, either by hand or using your favourite symbolic algebra package.

(b) From part (a), we know that $\log r$ is harmonic and its level sets are circles (boundaries of the annulus). Now we just shift and dilate $\log r$ so that we satisfy the boundary conditions:

$$V(r) = V_1 + \frac{(V_2 - V_1)}{\log(V_2/V_1)} \log\left(\frac{r}{V_1}\right).$$

(c) From part (a), we know that $\theta(x, y)$ is harmonic and has rays (boundaries of the wedge) as level sets. Thus, we could repeat the construction in (b), with θ instead of $\log r$, to solve this boundary value problem.

8. **Whittaker-Shannon sampling theorem.***

(a) To do the Fourier transform, we first recall the definition of $Ш_T$ and its Fourier series rerpresentation:

$$Ш_T(x) = \sum_{k=-\infty}^{\infty} \delta(x - kT) = \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{i(2\pi/T)k\theta}.$$

[2]In more advanced applications we do need to be more careful, since there are *many* complex logarithms, depending on the range of θ we choose. Another way of saying the same thing is that the complex log is naturally multi-valued!

The Fourier transform is therefore

$$\begin{aligned}\hat{\mathcal{F}}[Ш_T](u) &= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(\theta - kT) e^{-2\pi i u x}\, dx \\ &= \sum_{k=-\infty}^{\infty} e^{-2\pi i u k T} \\ &= \frac{1}{T} Ш_{1/T}(u),\end{aligned}$$

where we used the Fourier transform representation of $Ш_{1/T}$ on the last line.

(b) From the convolution theorem,

$$\begin{aligned}\hat{\mathcal{F}}[f_T](u) = (\hat{\mathcal{F}}[Ш_T] * F)(u) &= \frac{1}{T}(Ш_{1/T} * F)(u) \\ &= \frac{1}{T} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \delta\left(\xi - \frac{k}{T}\right) F(u - \xi)\, d\xi \\ &= \frac{1}{T} \sum_{k=-\infty}^{\infty} F\left(u - \frac{k}{T}\right).\end{aligned}$$

(c) For a bandlimited function f of width L, if we take a bunch of copies of the Fourier transform and simply space them out at intervals greater than L, we will be able to isolate a *single* copy of F. But that is precisely what $\hat{\mathcal{F}}[f_T]$ allows us to do! The spacing of the copies is T^{-1}, so we must choose our sampling rate $T^{-1} \geq L$, or equivalently, our intervals $T \leq L^{-1}$. Thus, we can reconstruct f by sampling at a discrete set of points.

9. **Diagonalisation and eigenbases.***

(a) Recall that, for finite-dimensional vector spaces, an operator A is diagonal with respect to a basis $(\mathbf{b}_1, \ldots, \mathbf{b}_n)$, with diagonal entries $(d_1, \ldots, d_n)$, just in case it acts on an arbitrary linear combination as

$$A \sum_{i=1}^{n} \alpha_i \mathbf{b}_i = \sum_{i=1}^{n} d_i \alpha_i \mathbf{b}_i.$$

This is equivalent to the statement $A\mathbf{b}_i = d_i \mathbf{b}_i$. In other words, each basis vector $\mathbf{b}_i$ is an eigenvector of A with eigenvalue d_i. Now consider a function space with basis elements labelled by $\mathbb{R}$. If we have an eigenbasis $\{f_\lambda : \lambda \in \mathbb{R}\}$ of $\mathcal{L}$, with $\mathcal{L} f_\lambda = d_\lambda f_\lambda$, then

$$\mathcal{L} \int f_\lambda \, d\lambda = \int d_\lambda f_\lambda \, d\lambda.$$

This is analogous to the finite-dimensional case with an integral replacing the finite sum, so we say $\mathcal{L}$ is diagonal.

(b) A simple choice is the derivative operator $D \equiv d/du$:

$$D f_\lambda(u) = \frac{d}{du} e^{2\pi i \lambda u} = 2\pi i \lambda f_\lambda.$$

10. **3D Bravais lattices.***

(a) As in the 2D case, we will check the result for $\mathbf{a}_1$ and invoke symmetry for the rest. We recall that $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{a}) = 0$. Hence,

$$\begin{aligned}
\mathbf{a}_1 \cdot \mathbf{b}_1 &= \frac{\mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{a}_3)}{\mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{a}_3)} = 1 \\
\mathbf{a}_1 \cdot \mathbf{b}_2 &= \frac{\mathbf{a}_1 \cdot (\mathbf{a}_3 \times \mathbf{a}_1)}{\mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{a}_3)} = 0 \\
\mathbf{a}_1 \cdot \mathbf{b}_3 &= \frac{\mathbf{a}_1 \cdot (\mathbf{a}_1 \times \mathbf{a}_2)}{\mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{a}_3)} = 0.
\end{aligned}$$

The same calculations repeated twice show that $\mathbf{a}_i \cdot \mathbf{b}_j = \delta_{ij}$. As in the 2D case, it then follows that the lattice generated by the $\mathbf{b}_i$ is reciprocal to $\mathbf{R}$.

(b) First, note that the triple product $\mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{a}_3)$ is just the volume of the parallelepiped with sides $\mathbf{a}_i$. In this case, it is a cube of volume a^3. Since $\mathbf{a}_i = a\hat{\mathbf{x}}_i$, we have

$$\mathbf{b}_1 = \frac{1}{a}(\hat{\mathbf{y}} \times \hat{\mathbf{z}}) = \frac{1}{a}\hat{\mathbf{x}},$$

and similarly $\mathbf{b}_2 = \hat{\mathbf{x}}/a$, $\mathbf{b}_3 = \hat{\mathbf{y}}/a$. So the reciprocal lattice is simple cubic.

(c) This simple (if slightly messy) calculation is left to the reader. The take-home message is that reciprocal lattices can be qualitatively different from the original lattice!

PHYC20014 Physical Systems

Wave Theory and Fourier Analysis: Tutorial 5

Tutorial problems

1. **Paradigms in optics.** There are many different mathematical models used in optics. For the following models, (i) explain the main idea of the model, (ii) state its regime of validity, and (iii) identify an optical phenomenon it doesn't explain.

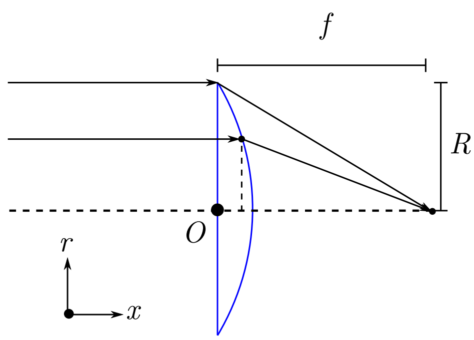
   (a) geometric optics;

   (b) Huygen's spherical wavelet model;

   (c) Fresnel-Huygens model;

   (d) classical electromagnetism.

2. **Thin lens.** Light passes through a thin, circular lens of radius R and is focussed at a point a distance f away. The refractive index of the lens is n. Consider light passing through the back of the lens at a vertical distance r from the centre. At this height, the lens has thickness $x(r)$. Assume that the lens is small ($f \gg R$) and thin ($r \gg x$).



   (a) Why must all rays take the same time to reach the focus? By considering a particular path, argue that

$$\frac{\sqrt{R^2 + f^2}}{c} = \frac{1}{c}\big[nx + \sqrt{r^2 + (f - x)^2}\big]. \tag{1}$$

   (b) Using (1) and the binomial approximation

$$\sqrt{1 + \epsilon} \approx 1 + \frac{1}{2}\epsilon, \quad |x| \ll 1,$$

   carefully show that the lens has a parabolic thickness profile

$$x = \frac{R^2 - r^2}{2(n-1)f}. \tag{2}$$

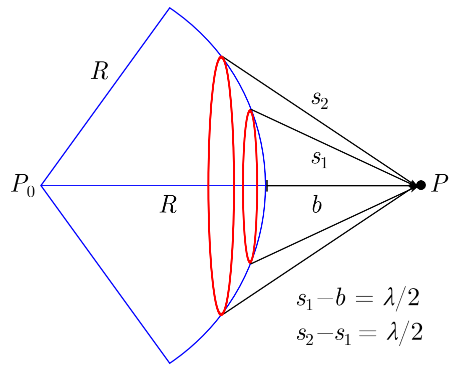
   (c) Sandwich together two thin lenses of radius R, with focal lengths f_1, f_2 and refractive index n. Using (2), derive the thin lens formula for the effective focal length f of the combined lens,

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}.$$

3. **Fresnel zones.** Consider a sphere of radius R centred at P_0, and a point P a distance b from the surface of the sphere. The nth *Fresnel zone* ($n = 1, 2, \ldots$) is the set of points on the sphere such that

$$b + \frac{(n-1)\lambda}{2} < s < b + \frac{n\lambda}{2},$$

where s is the distance from P to the point. Put another way, zone boundaries are defined by an optical path length difference (OPD) of $\lambda/2$.



(a) Explain in words why we expect destructive interference between adjacent Fresnel zones.

(b) Let N be the total number of Fresnel zones. Show that N satisfies

$$N \leq \frac{2}{\lambda}\left[\sqrt{b(2R+b)} - b\right] < N + 1.$$

(c) Let r_N denote inner radius of the largest zone as it appears to an observer at P. Show that r_N must obey

$$r_N < \frac{R}{R+b}\sqrt{b(2R+b)}.$$

4. **Plane waves and the Helmholtz equation.** Consider the plane wave

$$\psi(\mathbf{x}, t) = \exp\left[i(\mathbf{k}\cdot\mathbf{x} - \omega t)\right],$$

where $\mathbf{k}$ is the wavevector, ω the angular frequency, and $k \equiv |\mathbf{k}|$.

(a) Show that, for fixed t, $\psi(\mathbf{x}, t)$ is constant on planes normal to $\mathbf{k}$.

(b) By considering a point of constant phase, and assuming the waves travel in direction $\hat{\mathbf{k}}$, show that the plane waves travel with speed $v = \omega/k$.

(c) Verify that $\psi(\mathbf{x}, t)$ satisfies the *Helmholtz equation*

$$(\nabla^2 + k^2)\psi = 0.$$

(d) Using the foregoing (or otherwise), deduce that ψ satisfies the wave equation

$$\nabla^2\psi = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\psi.$$

Extra problems

5. **Fresnel lens.** A *Fresnel lens* is a means of constructing a large lens with a well-defined focal length f and maximum thickness d. We start with a circular lens of radius R_1, and construct concentric rings of outer radius R_k around the central lens.

   (a) Using (2), derive the recurrence relation for R_k:

   $$R_1 = \sqrt{2fd(n-1)}, \quad R_{k+1} = \sqrt{R_k^2 + 2fd(n-1)}.$$

   (b) Solve the recurrence and show that R_k scales as $\sqrt{k}$.

   (c) Using part (b), deduce that in the limit $k \to \infty$, the width $\Delta R_{k+1} \equiv R_{k+1} - R_k$ of the kth concentric ring obeys

   $$\Delta R_k \simeq \sqrt{\frac{fd(n-1)}{2k}}.$$

6. **Schuster's trick.** Schuster's trick is used to evaluate the alternating sum arising from the Fresnel zones. For a finite sequence $K_1, \ldots, K_N$, define

   $$\Sigma \equiv K_1 - K_2 + K_3 - K_4 + \ldots + (-1)^{N+1} K_N.$$

   Assume that N is even and $2K_n < K_{n-1} + K_{n+1}$ for $1 < n < N$.

   (a) Following lectures, show that

   $$\frac{1}{2}(K_1 - K_2) < \Sigma + \frac{1}{2}(K_N - K_1) < \frac{1}{2}(K_{N-1} - K_N).$$

   (b) From (a), infer that if $K_1 \approx K_2$ and $K_{N-1} \approx K_N$, then

   $$\Sigma \approx \frac{1}{2}(K_1 - K_N).$$

   (c) *For $K_n = n^2$, first check that $2K_n < K_{n-1} + K_{n+1}$. If $N = 2m$ is even, the exact (alternating) sum is

   $$\Sigma_{2m} = -m(2m+1).$$

   Show that as $m \to \infty$,

   $$\frac{\Sigma_{2m}}{\frac{1}{2}(K_1 - K_{2m})} \to 1.$$

   In asymptotic notation, this is written $\Sigma_{2m} \sim (K_1 - K_{2m})/2$. The error grows linearly with m, but this is much slower than the sums themselves!

7. **Inscribed polygons.** Take a circle, and pick n points on the boundary. Now join adjacent points to form an n-gon. Using mirrors, a laser, and Fermat's principle, show that the n-gon with minimal perimeter is the regular n-gon.

PHYC20014 Physical Systems

Wave Theory and Fourier Analysis: Tutorial 5

Solutions

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1. **Paradigms in optics.**

   (a) (i) Light consists of massless particles whose trajectory is governed by Fermat's principle.

   (ii) Geometric optics is valid for plane waves when the wavelength of light is much larger than the structures it interacts with.

   (iii) Diffraction. The assumption about wavelength breaks down, and the *wavelike* nature of light comes to the fore.

   (b) (i) Light is a wave disturbance; the envelope of the disturbance is called the *wavefront*, and each point on the wavefront is a source of secondary spherical waves, or *wavelets*.

   (ii) Huygens' model is valid when interference can be ignored.

   (iii) Interference! This requires information about the *phase* of the wave which Huygen's wavelet theory neglects.

   (c) (i) Light is a spreading wavefront with *phase* as well as amplitude.

   (ii) Valid when only the *scalar* aspect of the wave (amplitude and phase) is important, e.g. diffraction theory.

   (iii) Polarisaton, or any other phenomenon where the *vector* nature of light becomes important.

   (d) (i) Light is *electromagnetic radiation*: a solution to Maxwell's equation taking the form of orthogonal, self-propagating electric and magnetic fields.

   (ii) The *classical* interpretation of Maxwell's equations breaks down when quantum mechanics comes into play.

   (iii) The single photon double-slit experiment. We need to think of photons as individual quanta, and the intensity as a probability distribution, to explain this one.

2. **Thin lens.**

   (a) The light must interfere constructively at the focus. If the passage times are different, there will be phase differences leading to destructive interference. To find the common passage time, we consider a ray travelling from $r = R$ (the top of the lens) to the focus:

   $$T_R = \frac{\sqrt{R^2 + f^2}}{c}.$$

   Now for an arbitrary r. The distance travelled through the lens is x, and the distance in air is $\sqrt{r^2 + (f - x)^2}$. Since the speed in the lens is c/n, and the speed in air is c, the time is

   $$T_r = \frac{nx}{c} + \frac{1}{c}\sqrt{r^2 + (f - x)^2}.$$

   Since the passage time is common, $T_R = T_r$ and we have the desired identity (1).

(b) Moving all the terms to one side, we get

$$
\begin{aligned}
0 &= nx + \sqrt{r^2 + (f-x)^2} - \sqrt{R^2 + f^2} \\
&= nx + f\sqrt{\left(\frac{r}{f}\right)^2 + \left(1 - \frac{x}{f}\right)^2} - f\sqrt{\left(\frac{R}{f}\right)^2 + 1} \\
&= nx + f\sqrt{1 - \frac{2x}{f} + \frac{r^2 + x^2}{f^2}} - f\sqrt{1 + \left(\frac{R}{f}\right)^2}.
\end{aligned}
$$

The lens is small and thin, so $f \gg R \geq r \gg x(r)$. Using these approximations, and the binomial approximation for the square root:

$$
\begin{aligned}
0 &= nx + f\sqrt{1 - \frac{2x}{f} + \frac{r^2 + x^2}{f^2}} - f\sqrt{1 + \left(\frac{R}{f}\right)^2} \\
&\approx nx + f\sqrt{1 - \frac{2x}{f} + \frac{r^2}{f^2}} - f\sqrt{1 + \left(\frac{R}{f}\right)^2} \\
&\approx nx + f + \frac{f}{2}\left(\frac{r^2}{f^2} - \frac{2x}{f}\right) - f - \frac{f}{2}\left(\frac{R}{f}\right)^2 = (n-1)x + \frac{r^2 - R^2}{2f}.
\end{aligned}
$$

Rearranging, we obtain the parabolic profile

$$
x(r) = \frac{R^2 - r^2}{2(n-1)f}.
$$

(c) The trick is simply to note from (2) that $x = \alpha/f$, where α is a constant depending on R and n. The thickness of the combined lens is therefore

$$
\frac{1}{f} = \frac{x}{\alpha} = \frac{x_1 + x_2}{\alpha} = \frac{1}{f_1} + \frac{1}{f_2}.
$$

You might worry that we are treating a biconvex lens, curved on both sides, as a simple lens with the same thickness profile. For our idealised thin lenses, the difference is negligible.

3. **Fresnel zones.**

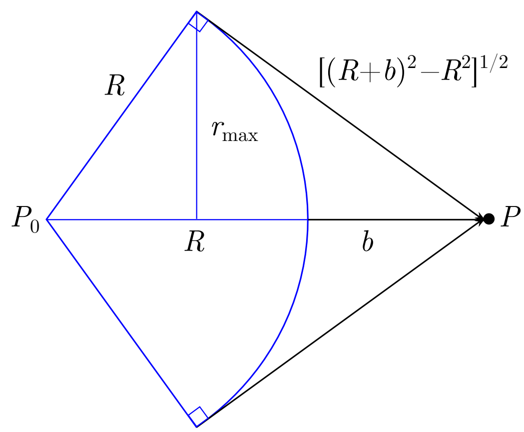
(a) By definition, rays from the *boundaries* of the Fresnel zones have a half-wavelength OPD and interfere destructively. Similarly, for each ray leaving a Fresnel zone, we can match it to a ray with half-wavelength OPD in either adjacent Fresnel zone. So in general, adjacent zones destructively intefere.

(b) The maximum OPD tells you the total number of Fresnel zones. The shortest path from P to the sphere has length b, while the longest (see the diagram) has length

$$
\sqrt{(R+b)^2 - R^2} = \sqrt{b(2R+b)}.
$$

The number of half-wavelengths ($\lambda/2$) contained in the maximum OPD is the total number of Fresnel zones N, so

$$
N \leq \frac{2}{\lambda}\left[\sqrt{b(2R+b)} - b\right] < N + 1.
$$



(c) To find the inner radius r_N, we must project the inner boundary of the last Fresnel zone onto a plane perpendicular to the line from P to P_0. Call the maximum projected radius $r_{\max}$; clearly, we must have $r_N < r_{\max}$. Using similar triangles (see diagram):

$$\frac{r_{\max}}{R} = \frac{\sqrt{b(2R+b)}}{R+b}.$$

Hence,

$$r_N < r_{\max} = \frac{R}{R+b}\sqrt{b(2R+b)}.$$

4. **Plane waves and the Helmholtz equation.**

(a) Recall from linear algebra that planes normal to a vector $\mathbf{k}$ are given by

$$\mathbf{k} \cdot \mathbf{x} = d$$

for some constant d. Thus, for fixed t, on these planes ψ is a constant:

$$\psi(\mathbf{x}, t) = \exp\left[i(\mathbf{k} \cdot \mathbf{x} - \omega t)\right] = \exp\left[i(d - \omega t)\right]$$

(b) To see how fast the waves move, follow a point of fixed phase as the wave moves over unit time. For instance, at $t = 0$, the phase at the origin is

$$i(\mathbf{k} \cdot \mathbf{x} - \omega t) = 0.$$

The wave propagates in direction $\hat{\mathbf{k}}$. At time $t = 1$, it has moved a distance v, where v is the speed. By setting the phase to zero (remember, we are trying to see how fast a point of constant phase moves), we can solve for v:

$$0 = i(\mathbf{k} \cdot \mathbf{x} - \omega t) = i(v\mathbf{k} \cdot \mathbf{k} - \omega) = i(vk - \omega).$$

Hence, $v = \omega/|\mathbf{k}|$.

(c) Since $\mathbf{k} \cdot \mathbf{x} = k_j x_j$, we have

$$\frac{\partial^2}{\partial x^2}\psi = (ik_1)^2\psi = -k_1^2\psi,$$

and similarly for y and z. It follows that

$$\nabla^2\psi = -(k_1^2 + k_2^2 + k_3^2)\psi = -k^2\psi.$$

Thus,

$$(\nabla^2 + k^2)\psi = 0.$$

(d) From the Helmholtz equation, $\nabla^2\psi = -k^2\psi$. However, twice taking the partial derivative with respect to time, we obtain

$$\ddot{\psi} = (i\omega)^2\psi = -\omega^2\psi.$$

Since $v = \omega/k$, we can assemble these two identities into a wave equation:

$$\nabla^2\psi = -k^2\psi = \frac{1}{v^2}(-\omega^2\psi) = \frac{1}{v^2}\ddot{\psi}.$$

5. **Fresnel lens.**

(a) See lecture notes.

(b) First of all, we guess that R_k (the outer radius) scales as $\sqrt{k}$, with $R_k = \sqrt{\alpha k}$ for some α. From R_1, we must have $\alpha = 2fd(n-1)$. Let's check if this choice satisfies the recurrence:

$$R_{k+1} = \sqrt{\alpha(k+1)} = \sqrt{\alpha k + \alpha} = \sqrt{R_k^2 + 2fd(n-1)}.$$

It does! So, as claimed, the outer radii scale as $\sqrt{k}$.

(c) From (b), the width of the kth annulus is

$$\Delta R_{k+1} = \sqrt{\alpha}(\sqrt{k+1} - \sqrt{k}) = \sqrt{\alpha}\frac{k+1-k}{\sqrt{k+1}+\sqrt{k}} = \frac{\sqrt{\alpha}}{\sqrt{k+1}+\sqrt{k}}.$$

In the limit $k \to \infty$, the difference between $k-1$ and k is negligible, so

$$\Delta R_k = \frac{\sqrt{\alpha}}{\sqrt{k}+\sqrt{k-1}} \approx \sqrt{\frac{fd(n-1)}{2k}}.$$

6. **Schuster's trick.**

(a) Since N is even and $2K_n < K_{n-1} + K_{n+1}$, from lecture notes we have

$$\Sigma < \frac{1}{2}K_1 + \frac{1}{2}K_1 - K_N$$
$$\Sigma > K_1 - \frac{1}{2}K_2 - \frac{1}{2}K_N.$$

We can rearrange to obtain the desired inequality:

$$\frac{1}{2}(K_1 - K_2) < \Sigma + \frac{1}{2}(K_N - K_1) < \frac{1}{2}(K_{N-1} - K_N).$$

(b) If $K_1 \approx K_2$ and $K_{N-1} \approx K_N$, then the LHS and RHS of the inequality in (b) are small. The middle term must also be small, or equivalently,

$$\Sigma \approx \frac{1}{2}(K_1 - K_N).$$

(c) First, we check our assumption about the growth of the sequence:

$$\begin{aligned} K_{n-1} + K_{n+1} - 2K_n &= (n-1)^2 + (n+1)^2 - 2n^2 \\ &= (2n^2 - 2n + 1) + (2n^2 + 2n + 1) - 2n^2 \\ &= 2 > 0. \end{aligned}$$

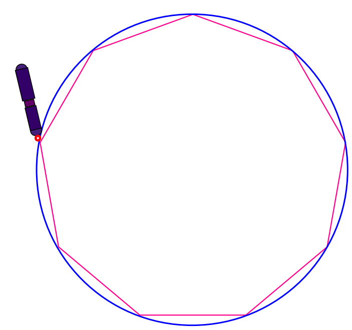
Hence, $K_{n-1} + K_{n+1} > 2K_n$ as required. Schuster's approximation for Σ_{2m} is

$$\frac{1}{2}(K_1 - K_{2m}) = \frac{1}{2}(1 - 4m^2).$$

As $m \to \infty$, the ratio is therefore

$$\frac{\Sigma_{2m}}{\frac{1}{2}(K_1 - K_{2m})} = \frac{4m^2 + 2m}{4m^2 - 1} = \frac{1 + \frac{1}{2m}}{1 - \frac{1}{4m^2}} \quad \to \quad 1.$$

7. **Inscribed polygons.** Suppose the boundary of the circle is reflective, and place it in a vacuum. Shoot a laser from one of the n vertices and demand it return to the same point (initial and final point specified).



By Fermat's principle, the path taken by light minimises the time; since it is travelling in vacuum, this is equivalent to minimising the perimeter of the shape described by the path. But since light travels in straight lines, and the angle of incidence equals the angle of reflection, the minimal perimeter n-gon is regular. (Technically, the argument also allows for *stellated* polygons, but these clearly have larger perimeters than the regular polygons.)

PHYC20014 Physical Systems

Wave Theory and Fourier Analysis: Tutorial 6

Tutorial problems

1. **Propagating plane waves.** Consider the spatial part of a plane wave of wavelength λ,

$$p(\mathbf{r}) = \exp(i\mathbf{k}\cdot\mathbf{r}) = \exp(2\pi i\mathbf{u}\cdot\mathbf{r}), \quad \mathbf{u} \equiv \frac{\mathbf{k}}{2\pi}.$$

Recall that $|\mathbf{k}| = 2\pi/\lambda$, so $|\mathbf{u}| = 1/\lambda$. Set $\mathbf{u}_\perp = (u_x, u_y)$ and $\mathbf{r}_\perp = (x, y)$.

(a) Show that

$$p(\mathbf{r}_\perp, z) = \exp(2\pi i\mathbf{u}_\perp\cdot\mathbf{r}_\perp)\exp\left[2\pi i\lambda^{-1}z\sqrt{1-\lambda^2u_\perp^2}\right].$$

(b) Take a slice of the wave at $z = 0$. Explain how lines of zero phase depend on $\mathbf{u}_\perp$.

(c) Let $\psi(\mathbf{r}_\perp, z)$ be a monochromatic wave packet, with λ as above, and $\Psi \equiv \hat{\mathcal{F}}[\psi(\mathbf{r}_\perp, 0)]$ the spatial Fourier transform of ψ in the $z = 0$ plane. Assume the wave is travelling in the z-direction, with $\Psi(\mathbf{u}_\perp) = 0$ for $\lambda u_\perp > 1$.

Since each plane-wave component evolves independently via (a), show that

$$\psi(\mathbf{r}_\perp, z) = \int d\mathbf{u}_\perp\,\Psi(\mathbf{u}_\perp)\exp(2\pi i\mathbf{u}_\perp\cdot\mathbf{r}_\perp)\exp\left[2\pi i\lambda^{-1}z\sqrt{1-\lambda^2u_\perp^2}\right].$$

2. **Fresnel and Fraunhofer.** Consider a monochromatic wavepacket as in Problem 1(b). If the *paraxial approximation* $u_\perp^2 \ll u^2$ holds for all nonzero spectral components, we can propagate the wavepacket foward in the z direction. To go from $z = z_1$ to $z = z_2$ (with $Z = z_2 - z_1$), we have

$$\psi(\mathbf{r}_\perp, z_2) = \int d\mathbf{r}'_\perp\,K(\mathbf{r}_\perp, z_2; \mathbf{r}'_\perp, z_1)\psi(\mathbf{r}'_\perp, z_1), \tag{1}$$

where K is the *Fresnel propagator*:

$$K(\mathbf{r}_\perp, z_2; \mathbf{r}'_\perp, z_1) \equiv \frac{1}{i\lambda Z}\exp(ikZ)\exp\left[\frac{i\pi(\mathbf{r}_\perp - \mathbf{r}'_\perp)^2}{\lambda Z}\right].$$

We also denote $\mathbf{r}_\perp = (x, y)$ and $\mathbf{r}'_\perp = (x', y')$.

(a) Express Fresnel propagation as a 2D convolution.

(b) Consider the *far-field regime* (aka *Fraunhofer approximation*), where the Fresnel integral (1) vanishes unless

$$|\mathbf{r}'_\perp|^2 \ll \lambda Z.$$

Show that, in this case, the Fresnel integral reduces to a Fourier transform with phase factors:

$$\psi(\mathbf{r}_\perp, z_2) = \frac{\exp(ikZ)}{i\lambda Z}\exp\left[\frac{i\pi|\mathbf{r}_\perp|^2}{\lambda Z}\right]\hat{\mathcal{F}}[\psi(\mathbf{r}'_\perp, z_1)](u_x, u_y)$$

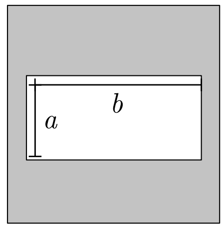
where $u_x \equiv x/\lambda Z$ and $u_y \equiv y/\lambda Z$. When we calculate diffraction patterns, we only want the *intensity* $|\psi|^2$ and the phase factors go away.

3. **Far-field rectangles.** Obstructions in the object plane at $z = 0$ form far-field diffraction (i.e. intensity) patterns when illuminated by a monochromatic plane wave of wavelength λ and amplitude A. Find the exact form of the diffraction pattern at $z = Z$ for the following apertures:

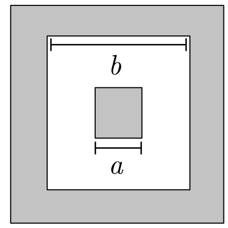
   (a) a rectangle with dimensions $a \times b$;

   (b) an opaque square of side length a in the middle of a square hole of side length b ($b > a$).

(a)



(b)



4. **Simple image processing.** You can *process* or *filter* a monochrome image in an analogue way as follows: (1) put a transparency behind a lens, (2) illuminate both with a plane wave, (3) apply a filter in the focal plane $Z = f$, then (4) use another lens at $Z = 2f$ to recover a flipped, processed version of the image. (Hopefully you remember this from second year labs.) Match the processed image of Einstein in the first column to the filter in the second:

Extra problems

5. **Lenses.** Position an object in front a lens and illuminate the lens and object from behind with a plane wave. The wave passes through the lens, then the object, and creates a diffraction pattern.

   (a) Without the lens, where would the rays focus?

   (b) Using part (a) and Problem 2(b), give a heuristic argument that the diffraction pattern in the *focal plane* is simply related to the Fourier transform of the light emerging from the object. In other words, a lens is an analogue Fourier transformer!

   (c) *It can be shown that light passing through a lens with focal length f undergoes a phase transformation
   $$T_l(\mathbf{r}'_\perp) = \exp\left[-\frac{i\pi|\mathbf{r}'_\perp|^2}{\lambda f}\right].$$
   Given this fact, demonstrate that Fresnel diffraction for the combined transmittance $T_l\psi$ is equal to far-field diffraction for ψ when $Z = f$. This is a rigorous version of (b)!

6. **Array theorem.** Suppose we take an opaque screen at $z = 0$, and several identically shaped, non-overlapping holes called an *array*. The amplitude transmittance for an individual hole is $T(x, y)$, and the holes are centred at $\mathbf{x}_j = (x_j, y_j)$ for $j = 1, \ldots, n$.

   (a) Write the transmittance T_A of the array by convolving T with a sum of delta functions.

   (b) Using properties of Fourier transforms (Tutorial 3), show that
   $$\hat{\mathcal{F}}[T_A](u_x, u_y) = \hat{\mathcal{F}}[T](u_x, u_y)\sum_{j=1}^{n} e^{-2\pi i(x_j u_x + y_j u_y)}.$$
   Applied to far-field diffraction, this result is called the *array theorem*.

   (c) *Apply the array theorem to find the far-field diffraction pattern due to n slits of width a and infinite height, uniformly spaced at intervals $d > a$.

7. **Babinet's principle.** Let A be an obstruction in the object plane at $z = 0$, and B the complementary obstruction. (Think of cutting A out of an infinite opaque sheet, leaving B.) Babinet's principle states that, for $\mathbf{u}_\perp \neq \mathbf{0}$, the Fourier transform of the transmittance functions F_A, F_B satisfies
   $$F_A(\mathbf{u}_\perp) = -F_B(\mathbf{u}_\perp).$$

   (a) A human hair is placed on a glass plate and illuminated with a handheld laser ($\lambda =$ 671 nm). A diffraction pattern (stripes parallel to the hair) is observed on a screen 1 m away, with the first minimum a distance 6.7 mm from the centre of the pattern. Use Babinet's principle to estimate the width of the hair.

   (b) Generalise Babinet's principle to a set of non-overlapping apertures A_i, $i = 1, \ldots, n$ whose union covers the image plane.

8. **Paraxial wave equation.** For a wave propagating in the z-direction (with only weak divergence), it's natural to factor it into a plane wave moving in the z direction and a *planar envelope function* controlling variation in the transverse plane:

$$\psi(\mathbf{r}) = \tilde{\psi}(\mathbf{r}_\perp, z)e^{ikz}.$$

The Helmholtz equation for ψ is

$$(\nabla^2 + k^2)\psi = 0.$$

Combining the two equations, and assuming that the planar envelope is a slowly-varying function of z ($|\partial_z^2\tilde{\psi}| \ll k|\partial_z\tilde{\psi}|$), we obtain the *paraxial wave equation* for $\tilde{\psi}$:

$$\left[\nabla_\perp^2 + 2ik\frac{\partial}{\partial z}\right]\tilde{\psi} = 0. \tag{2}$$

(a) Show that, formally speaking, (2) can be written as a 2D *diffusion equation* with diffusion coefficient $D = 1/2k$ and imaginary time $t = iz$.

(b) Conclude that the fundamental solution to (2) is

$$\tilde{\psi}(\mathbf{r}_\perp, z) = \frac{k}{2\pi iz}\exp\left[\frac{ik|\mathbf{r}_\perp|^2}{2z}\right].$$

(c) On Level 6 of the physics building, Professor Quiney does imaging with X-rays and Professor Allen does imaging with electrons. The two have a long-running dispute about which is superior. In fact, the different methods of imaging are governed by the same mathematics! Show that the paraxial wave equation is identical to the Schrödinger equation for a free particle in 2D, with $z = t$ and mass $m = \hbar k$.

9. **Circular holes and Airy patterns.*** Consider the diffraction setup of Problem 3 with a *circular* aperture of radius R. To find the far-field diffraction pattern, we need to Fourier transform the transmittance function in radial coordinates:

$$f(r) = \begin{cases} 1 & 0 \le r \le R \\ 0 & R < r. \end{cases}$$

(a) Write the Fourier integral in polar coordinates, using (r, θ) in the object plane and (ρ, ϕ) in Fourier space. You should find

$$F(\rho, \phi) = \int_0^\infty dr\, r f(r) \int_0^{2\pi} d\theta\, \exp(-2\pi i r\rho\cos(\theta - \phi)).$$

(b) Using the Bessel function identities

$$J_0(\beta) = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, \exp(-i\beta\theta)$$
$$\beta J_1(\beta) = \int_0^\beta sJ_0(s)\, ds,$$

simplify the result in (a) to obtain

$$F(\rho, \theta) = \frac{R}{\rho}J_1(2\pi\rho R).$$

(c) Conclude that the far-field diffraction pattern at $z = Z$ is

$$I(r) = 4I_0\pi^2 R^4 \left[\frac{J_1(\pi\kappa)}{\pi\kappa}\right]^2, \quad \kappa \equiv \frac{2Rr}{\lambda Z}$$

where $I_0 = A^2$ is, as usual, the intensity of the illuminating plane wave. This is called the *Airy pattern.*

(d) The function $[J_1(\pi\kappa)/(\pi\kappa)]^2$ has a minimum at $\kappa = 1.22$. Argue that the width d of the central lobe in the Airy pattern in (c) is therefore

$$d = 1.22\,\frac{\lambda Z}{R}.$$

This is the basis of the *Rayleigh criterion* for the resolution of an optical system.

10. **Hankel transforms.*** Suppose $g(r)$ is a function which only depends on radial distance from the origin. The m-th order *Hankel transform* of $g(r)$ is like a Fourier transform, but weighted by the m-th Bessel function of the first kind J_m instead of a plane wave:

$$\hat{\mathcal{H}}_m[g(r)](\rho) = 2\pi \int_0^\infty dr\, r g(r) J_m(2\pi r\rho).$$

You can regard the functions J_m as "black boxes" whose defining properties we will specify as we need. For separable functions in polar coordinates, we can express the Fourier transform in terms of Hankel transforms.

(a) Suppose that in polar coordinates, the function $g(x, y)$ takes the specific form

$$g(r, \theta) = R(r)e^{im\theta}, \quad m \in \mathbb{Z}.$$

Show that the Fourier transform $G = \hat{\mathcal{F}}[g]$ may be written in terms of the Hankel transform as

$$G(\rho, \phi) = (-i)^m e^{im\phi} \hat{\mathcal{H}}_m[R](\rho),$$

where (ρ, ϕ) are polar coordinates in Fourier space. You will need the following identity for Bessel functions:

$$\exp(ia\sin\phi) = \sum_{k=-\infty}^{\infty} J_k(a)e^{ik\phi}.$$

(b) Now consider an arbitrary separable function $g(r, \theta) = R(r)\Theta(\theta)$. Use part (a) to show that,

$$G(\rho, \phi) = \sum_{k\in\mathbb{Z}} c_k(-i)^k e^{ik\phi} \hat{\mathcal{H}}_k[R](\rho),$$

for

$$c_k = \int_0^{2\pi} d\theta\, e^{-ik\theta}\Theta(\theta).$$

PHYC20014 Physical Systems

Wave Theory and Fourier Analysis: Tutorial 6

Solutions

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1. **Propagating plane waves.**

   (a) We have

   $$u^2 = \lambda^{-2} = u_\perp^2 + u_z^2 \quad \Longrightarrow \quad u_z = u\sqrt{1 - \lambda^2 u_\perp^2}.$$

   Hence,

   $$\begin{aligned} \exp(2\pi i \mathbf{u} \cdot \mathbf{r}) &= \exp(2\pi i \mathbf{u}_\perp \cdot \mathbf{r}_\perp) \exp(2\pi i u_z z) \\ &= \exp(2\pi i \mathbf{u}_\perp \cdot \mathbf{r}_\perp) \exp\left[2\pi i \lambda^{-1} z \sqrt{1 - \lambda^2 u_\perp^2}\right] \end{aligned}$$

   as required.

   (b) At $z = 0$, lines of zero phase correspond to $\mathbf{u}_\perp \cdot \mathbf{r}_\perp = u_x x + u_y y = n$ for integer n. These lines have slope u_x/u_y and separation $u_\perp^{-1}$.

   (c) From the definition of Ψ,

   $$\psi(\mathbf{r}_\perp, 0) = \int d\mathbf{u}_\perp \, \Psi(\mathbf{u}_\perp) \exp(2\pi i \mathbf{u}_\perp \cdot \mathbf{r}_\perp).$$

   Using part (a) to propagate each plane wave component forward, and the assumption that $\Psi(\mathbf{u}_\perp) = 0$ for $\lambda u_\perp > 1$, we deduce that

   $$\psi(\mathbf{r}_\perp, z) = \int d\mathbf{u}_\perp \, \Psi(\mathbf{u}_\perp) \exp(2\pi i \mathbf{u}_\perp \cdot \mathbf{r}_\perp) \exp\left[2\pi i \lambda^{-1} z \sqrt{1 - \lambda^2 u_\perp^2}\right].$$

2. **Fresnel and Fraunhofer.**

   (a) First, we note that the Fresnel kernel K only depends on $\mathbf{r}_\perp - \mathbf{r}'_\perp$. Hence, we can define

   $$\psi_z(\mathbf{r}_\perp) \equiv \psi(\mathbf{r}_\perp, z), \qquad K_{z_2,z_1}(\mathbf{r}_\perp - \mathbf{r}'_\perp) \equiv K(\mathbf{r}_\perp, z_2; \mathbf{r}'_\perp, z_1).$$

   It follows that

   $$\begin{aligned} \psi(\mathbf{r}_\perp, z_2) &= \int d\mathbf{r}'_\perp \, K_{z_2,z_1}(\mathbf{r}_\perp - \mathbf{r}'_\perp) \psi_{z_1}(\mathbf{r}'_\perp) \\ &= (K_{z_2,z_1} * \psi_{z_1})(\mathbf{r}_\perp). \end{aligned}$$

   In fact, the Fresnel propagator is a *Green's function*, as encountered in our discussion of PDEs earlier in the course. In optics, it is called a *point-spread function* rather than a Green's function.

(b) First, we need to simplify the Fresnel propagator using the far-field assumption:

$$\begin{aligned}
K(\mathbf{r}_\perp, z_2; \mathbf{r}'_\perp, z_1) &= \frac{1}{i\lambda Z}\exp(ikZ)\exp\left[\frac{i\pi(\mathbf{r}_\perp - \mathbf{r}'_\perp)^2}{\lambda Z}\right] \\
&= \frac{1}{i\lambda Z}\exp(ikZ)\exp\left[\frac{i\pi|\mathbf{r}_\perp|^2 + |\mathbf{r}'_\perp|^2 - 2\mathbf{r}_\perp\cdot\mathbf{r}'_\perp}{\lambda Z}\right] \\
&= \frac{1}{i\lambda Z}\exp(ikZ)\exp\left[\frac{i\pi|\mathbf{r}_\perp|^2}{\lambda Z}\right]\exp\left[\frac{i\pi|\mathbf{r}'_\perp|^2}{\lambda Z}\right]\exp\left[\frac{-2\pi i(xx' + yy')}{\lambda Z}\right] \\
&\approx \frac{1}{i\lambda Z}\exp(ikZ)\exp\left[\frac{i\pi|\mathbf{r}_\perp|^2}{\lambda Z}\right]\exp\left[-2\pi i(u_x x' + u_y y')\right]
\end{aligned}$$

where $u_x \equiv x/\lambda Z$ and $u_y \equiv y/\lambda Z$. Substituting into (1), we obtain

$$\begin{aligned}
\psi(\mathbf{r}_\perp, z_2) &\approx \frac{1}{i\lambda Z}\exp(ikZ)\exp\left[\frac{i\pi|\mathbf{r}_\perp|^2}{\lambda Z}\right]\int d\mathbf{r}'_\perp\,\psi(\mathbf{r}'_\perp, z_1)\exp\left[-2\pi i(u_x x + u_y y)\right] \\
&= \frac{1}{i\lambda Z}\exp(ikZ)\exp\left[\frac{i\pi|\mathbf{r}_\perp|^2}{\lambda Z}\right]\hat{\mathcal{F}}[\psi(\mathbf{r}'_\perp, z_1)](u_x, u_y).
\end{aligned}$$

3. **Far-field rectangles.**

(a) In this case, the transmittance T_1 is just a product of rect functions, say

$$T_1(x, y) = \mathrm{rect}\left(\frac{x}{a}\right)\mathrm{rect}\left(\frac{y}{b}\right).$$

The far-field amplitude is proportional to the Fourier transform $\hat{\mathcal{F}}[T_1]$. Since the function is separable in Cartesian coordinates, the Fourier transform factors into separate transforms in x and y. Finally, we use the Fourier transform of the rect and the similarity theorem:

$$\begin{aligned}
\hat{\mathcal{F}}[T_1](u_x, u_y) &= \hat{\mathcal{F}}\left[\mathrm{rect}\left(\frac{x}{a}\right)\right](u_x)\hat{\mathcal{F}}\left[\mathrm{rect}\left(\frac{y}{b}\right)\right](u_y) \\
&= ab\,\mathrm{sinc}(au_x)\mathrm{sinc}(bu_y).
\end{aligned}$$

Hence, the diffraction pattern in the image plane at $z = Z$ is

$$I(x, y) = I_0|\hat{\mathcal{F}}[T_1](x/\lambda Z, y/\lambda Z)|^2 = \frac{I_0 x^2 y^2}{(\lambda Z)^4}\,\mathrm{sinc}^2\left(\frac{ax}{\lambda Z}\right)\mathrm{sinc}^2\left(\frac{by}{\lambda Z}\right)$$

where $I_0 = A^2$ is the intensity of the illuminating plane wave. There is a cartoon of the pattern below, courtesy of Mathematica:

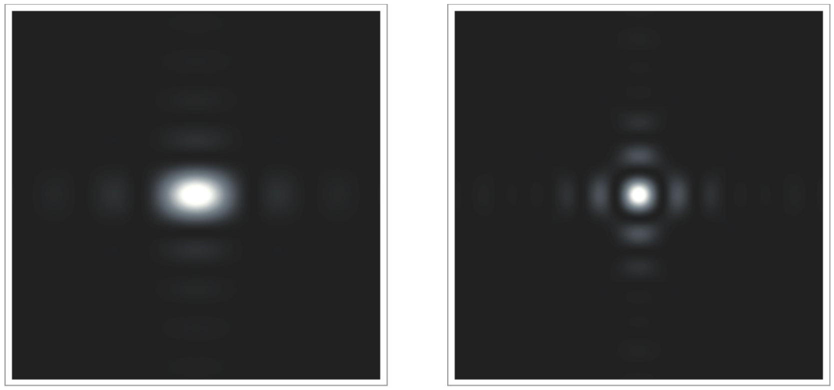
(b) The transmittance function is now

$$T_2(x, y) = \mathrm{rect}\left(\frac{x}{b}\right)\mathrm{rect}\left(\frac{y}{b}\right) - \mathrm{rect}\left(\frac{x}{a}\right)\mathrm{rect}\left(\frac{y}{a}\right).$$

We can use part (a) and the linearity of the Fourier transform:

$$\hat{\mathcal{F}}[T_2](u_x, u_y) = b^2\,\mathrm{sinc}(bu_x)\mathrm{sinc}(bu_y) - a^2\,\mathrm{sinc}(au_x)\mathrm{sinc}(au_y).$$

I leave squaring this to your imagination. Again, we draw a cartoon version below.



4. **Simple image processing.** $1 \to 2$: The slit mask allows through spectral information about vertical frequency, and filters out almost everything else. That is why the image is blurred in the horizontal directions; most of that information has been lost.

   $2 \to 3$: This is a *low-pass filter*. It gets rid of high frequency information, leaving only blurry, slowly changing spectral components.

   $3 \to 1$: This is a *high-pass filter*. It gets rid of low frequency components (large patches of colour), leaving only rapidly changing parts of the image. For this reason, high-pass filters are used for *edge detection*.

5. **Lenses.**

   (a) At "infinity", since they remain parallel forever.

   (b) Infinitely far away, we would get far-field, Fraunhofer diffraction. Since the lens effectively relocates infinity to the focal plane, it stands to reason that the diffraction pattern formed there is Fraunhofer.

   (c) *Going through the argument in Problem 2(b), we see that all we need to do to change Fresnel to Fraunhofer diffraction is get rid of the factor

   $$T_l(\mathbf{r}'_\perp) = \exp\left[\frac{i\pi|\mathbf{r}'_\perp|^2}{\lambda Z}\right]$$

   in the Fresnel propagator $K(\mathbf{r}_\perp, z_2; \mathbf{r}'_\perp, z_1)$. The far-field approximation is one way to do it, but the phase factor from the lens also works perfectly! See the lecture notes for further details.

6. **Array theorem.**

   (a) Convolving with a delta function $\delta(x - x_0)\delta(y - y_0)$ simply shifts a function to (x_0, y_0), so convolving with a sum S of appropriately centred deltas will give us a sum of shifted copies:

   $$T_A = T * S, \quad S(\mathbf{r}) \equiv \sum_{j=1}^{n} \delta(\mathbf{r} - \mathbf{x}_j).$$

(b) Using the convolution and shift theorems from Tutorial 3,

$$\begin{aligned}
\hat{\mathcal{F}}[T_A](u_x,u_y) &= \hat{\mathcal{F}}[T*S](u_x,u_y) = \hat{\mathcal{F}}[T](u_x,u_y)\hat{\mathcal{F}}[S](u_x,u_y) \\
&= \hat{\mathcal{F}}[T](u_x,u_y)\sum_{j=1}^{n}\hat{\mathcal{F}}[\delta(\mathbf{r}-\mathbf{x}_j)](u_x,u_y) \\
&= \hat{\mathcal{F}}[T](u_x,u_y)\sum_{j=1}^{n} e^{-2\pi i(x_j u_x + y_j u_y)}.
\end{aligned}$$

(c) Orient the slits in the y-direction. The slit transmittance is just a rect function:

$$T(x,y) = \mathrm{rect}\left(\frac{x}{a}\right).$$

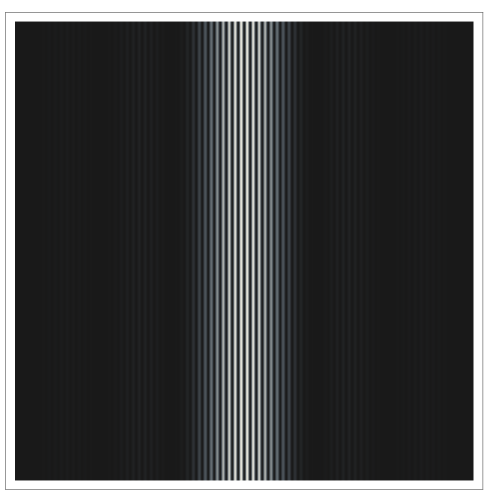
Write the position of the slits as $\mathbf{x}_j = (dj, 0)$ for $j = 1,\ldots,n$, though the result will be the same if we shift these up or down in the y-direction a fixed amount. Thus, from the array theorem and our results for the rect function, the relevant Fourier transform is

$$\begin{aligned}
\hat{\mathcal{F}}[T](u_x,u_y)\sum_{j=1}^{n} e^{-2\pi i(x_j u_x + y_j u_y)} &= a\,\mathrm{sinc}(au_x)\sum_{j=1}^{n}\left(e^{-2\pi i d u_x}\right)^j \\
&= a\,\mathrm{sinc}(au_x)\frac{z^{(n+1)/2}}{z^{-1/2}}\frac{z^{-(n+1)/2} - z^{(n+1)/2}}{z^{-1/2} - z^{1/2}} \qquad z \equiv e^{-2\pi i d u_x} \\
&= a\, e^{-\pi(n+2)i d u_x}\,\mathrm{sinc}(au_x)\frac{\sin[\pi d u_x (n+1)]}{\sin[\pi d u_x]}.
\end{aligned}$$

Thus, the diffraction pattern is

$$I(x) = I_0 a^2\,\mathrm{sinc}^2\left(\frac{ax}{\lambda Z}\right)\frac{\sin^2[\pi d x(n+1)/\lambda Z]}{\sin^2[\pi d x/\lambda Z]}.$$

This is the pattern from an idealised n slit diffraction, with a sinc envelope due to the finite width of individual slits. For instance, here is the pattern for $d = 2a$, $n = 5$:



7. **Babinet's principle.**

(a) Let w be the width of the hair. From Babinet's principle, the far-field diffraction pattern from the hair is practically identical to the pattern from an aperture of the same width. Using calculations from lectures or Problem 2(a), the diffraction pattern of a slit of width w is proportional to $\mathrm{sinc}^2(wx/\lambda L)$. Since the first zero of $\mathrm{sinc}(t)$ occurs $t = 1$, and the first minimum in the diffraction pattern occurs at $x = 6.7$ mm, we have

$$\frac{wx}{\lambda L} = 1 \quad \Longrightarrow \quad w = \frac{\lambda L}{x} = \frac{671 \text{ nm} \times 1 \text{ m}}{6.7 \text{ mm}} \approx 100\,\mu\text{m}.$$

If you have a laser pointer handy, you can actually try this at home!

(b) Suppose each aperture A_i has transmittance T_i. By definition, the combined transmittance of the apertures is always unity, since at each point, it is 1 over some aperture A_i and zero elsewhere:

$$\sum_{i=1}^{n} T_i = 1.$$

Now split an incoming wave ψ into patches emerging from each aperture:

$$\psi = \sum_{i=1}^{n} T_i \psi = \sum_{i=1}^{n} \psi_i.$$

A simple generalisation of the argument from lectures shows that

$$\sum_{i=1}^{n} \hat{\mathcal{F}}[T_i](\mathbf{u}_\perp) = 0, \quad \mathbf{u}_\perp \neq \mathbf{0}.$$

8. **Paraxial wave equation.**

(a) The diffusion equation in 3D is

$$\frac{\partial \tilde{\psi}}{\partial t} = D\nabla_\perp^2 \tilde{\psi}.$$

Making the substitutions $D = 1/2k$ and $t = -iz$, we recover the paraxial wave equation:

$$2ik\frac{\partial \tilde{\psi}}{\partial z} = \nabla_\perp^2 \tilde{\psi}.$$

(b) We simply substitute $D = 1/2k$ and $t = -iz$ into the fundamental solution for the wave equation in 2D,

$$\Phi(x, y, t) = \frac{1}{4\pi Dt} \exp\left[-\frac{x^2 + y^2}{4Dt}\right].$$

(c) This is a similar exercise to (a). The 2D Schrödinger equation for a free particle of mass m is

$$i\hbar\frac{\partial}{\partial t}\tilde{\psi} = -\frac{\hbar^2}{2m}\nabla_\perp^2 \tilde{\psi}.$$

After a little algebra, we see this is identical to the paraxial wave equation for $m = \hbar k$ and $z = t$.

9. **Circular holes and Airy patterns.***

(a) First, convert the polar coordinates in Fourier space (ρ, ϕ) to Cartesian coordinates:

$$u_x = \rho \cos\phi, \quad u_y = \rho \sin\phi.$$

Now we just plug these into the usual definition, change to polar coordinates

$$x = r\cos\theta, \quad y = r\sin\theta,$$

and use a double angle formula:

$$\begin{aligned}
F(\rho,\phi) &= \hat{\mathcal{F}}[f](\rho\cos\phi, \rho\sin\phi) \\
&= \int_{-\infty}^{\infty} dx\, dy\, f(\sqrt{x^2+y^2}) \exp\left[-2\pi i\rho(x\cos\phi + y\sin\phi)\right] \\
&= \int_0^{\infty} dr\, r f(r) \int_0^{2\pi} d\theta\, \exp\left[-2\pi i\rho r(\cos\theta\cos\phi + \sin\theta\sin\phi)\right] \\
&= \int_0^{\infty} dr\, r f(r) \int_0^{2\pi} d\theta\, \exp\left[-2\pi i\rho r\cos(\theta - \phi)\right].
\end{aligned}$$

(b) Making the change of variables $\vartheta = \theta - \phi$ in the θ integral (and exploiting the periodicity of cosine), followed by the first Bessel function identity, we get

$$\begin{aligned}
F(\rho,\phi) &= \int_0^{\infty} dr\, r f(r) \int_0^{2\pi} d\vartheta\, \exp\left[-2\pi i\rho r\cos\vartheta\right] \\
&= 2\pi \int_0^{\infty} dr\, r f(r) J_0(2\pi\rho r) \\
&= 2\pi \int_0^{R} dr\, r J_0(2\pi\rho r).
\end{aligned}$$

We can make the change of variable $s = 2\pi\rho r$ and use the second Bessel function identity:

$$F(\rho,\phi) = \frac{1}{2\pi\rho^2}\int_0^{2\pi\rho R} ds\, s J_0(s) = \frac{R}{\rho} J_1(2\pi\rho R).$$

(c) Under the usual substitutions $u_x = x/\lambda Z$, $u_y = y/\lambda Z$, the polar variable $\rho = r/\lambda Z$. Now we substitute into the result from (b) and square to get the intensity of the far-field diffraction pattern:

$$\begin{aligned}
I(r) &= I_0 |F(r/\lambda Z)|^2 \\
&= I_0 \left| 2\pi R^2 \frac{J_1(\pi\kappa)}{\pi\kappa} \right|^2 \\
&= 4 I_0 \pi^2 R^4 \left[\frac{J_1(\pi\kappa)}{\pi\kappa}\right]^2,
\end{aligned}$$

where $\kappa \equiv 2Rr/\lambda Z$. Here is a graph of the Airy pattern:

(d) The width d of the central lobe is given by

$$1.22 = \kappa = \frac{2Rr}{\lambda Z} \quad \Longrightarrow \quad d = 2r = 1.22\,\frac{\lambda Z}{R}.$$

10. **Hankel transforms.***

(a) Using the same strategy as Problem 9(a):

$$\begin{aligned} G(\rho,\phi) &= \int_0^\infty dr\, rR(r) \int_0^{2\pi} d\theta\, e^{im\theta} \exp\left[-2\pi i \rho r \cos(\theta-\phi)\right] \\ &= \int_0^\infty dr\, rR(r) \int_0^{2\pi} d\theta\, e^{im\theta} \exp\left[2\pi i \rho r \sin(\phi-\theta-\pi/2)\right]. \end{aligned}$$

Usung the identity for Bessel functions, we obtain

$$\begin{aligned} G(\rho,\phi) &= \int_0^\infty dr\, rR(r) \int_0^{2\pi} d\theta\, e^{im\theta} \sum_{k=-\infty}^{\infty} J_k(2\pi\rho r) \exp[ik(\phi-\theta-\pi/2)] \\ &= \sum_{k=-\infty}^{\infty} (-i)^k e^{ik\phi} \int_0^\infty dr\, r J_k(2\pi\rho r) R(r) \int_0^{2\pi} d\theta\, e^{i(m-k)\theta} \\ &= \sum_{k=-\infty}^{\infty} (-i)^k e^{ik\phi} \delta_{mk} \cdot 2\pi \int_0^\infty dr\, r J_k(2\pi\rho r) R(r) \\ &= (-i)^m e^{im\phi} \hat{\mathcal{H}}_m[R](\rho). \end{aligned}$$

Along the way, we used the fact that $e^{-i\pi/2} = -i$ and

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta\, e^{i(m-k)\theta} = \delta_{mk}.$$

(b) Since $\Theta(\theta)$ is a function of polar angle, it is periodic with period $T = 2\pi$. Hence, it has an exponential Fourier series

$$\Theta(\theta) = \sum_{k=-\infty}^{\infty} c_k e^{ik\theta}.$$

Thus, we can write

$$g(r,\theta) = \sum_{k=-\infty}^{\infty} c_k e^{ik\theta} R(r).$$

Using the linearity of the Fourier transform and part (a), we obtain

$$\begin{aligned} G(\rho,\phi) = \hat{\mathcal{F}}[g](\rho,\phi) &= \sum_{k=-\infty}^{\infty} c_k \hat{\mathcal{F}}[e^{ik\theta} R(r)](\rho,\phi) \\ &= \sum_{k=-\infty}^{\infty} c_k (-i)^k e^{ik\phi} \hat{\mathcal{H}}_k[R](\rho). \end{aligned}$$

Plugging in the definition for the exponential Fourier coefficients c_k of $\Theta(\theta)$ gives us the final result.