Black hole mergers

LIGO detects energetic bursts of gravity waves released when massive objects, like black holes or neutron stars, collide. Neutron stars are tricksy characters, complicated and poorly-understood. Black holes, in contrast, are the simplest objects in the universe. In the words of Nobel-winning astrophysicist Subrahmanyan Chandrasekhar,

> The black holes of nature are the most perfect macroscopic objects there are in the universe: the only elements in their construction are our concepts of space and time.

They are much, much simpler than your cat. Maybe it's not so crazy to try and estimate the energy released in a black hole merger!

Black holes. Our goal will be to think about energy, nothing fancy to do with the form of the gravity waves (though we might say more about that below).

- First of all, what is a black hole? What parameters can it be governed by?
- Everyone should know this hopefully: black holes are regions where gravity is strong enough to trap light. Since nothing can travel faster than light, *nothing* can escape!
- They might list radius, temperature, etc; we will explain that these things actually depend on the other parameters!
- Since black holes are made out of gravity, the most important parameter is the mass M. It turns out that black holes can spin and become charged (e.g. if it sucks up some hydrogen ions), so angular momentum J and electric charge Q are also relevant.
- (We can talk about conserved quantities, spacetime, etc.)

Black holes which have had a chance to settle down are characterised by only three numbers: mass M, spin J, and charge Q. This is called the no-hair theorem. Black holes undergoing violent perturbations are definitely not settled, so there are many, many more things that can be going on. (Still much simpler than perturbations to your cat.) To make our lives easy, whenever a black hole is having a rough time, let's just ignore it until it settles down again. Agreed? So "black hole" will mean "settled black hole" unless we say otherwise.

- Black holes in nature are usually neutral or very close to neutral, with $Q = 0$. Why?
- Electromagnetism is much, much stronger than gravity. Charged black holes will almost immediately suck up nearby opposite charges to become neutral. Also, collapsed matter (how black holes actually form!) is usually neutral.
- (If students seem adventurous.) We can measure the relative strength of gravity and electromagnetism but putting two electrons next to each. What is the ratio of the strength of electrostatic repulsion to gravitational attraction?
- They will need some numbers and maybe some laws. The end result is about 10^43!

- If gravity is so much weaker than electromagnetism, why is the shape of the universe dictated by gravity?
- Precisely *because* it is weak! Any big charge imbalance leads to massive instabilities which (usually) quickly neutralise. Put another way, electromagnetism *acts fast*. Gravity is weak and acts slowly. But evidently, slow and steady wins the race.

To make our lives even easier, let's focus on the simplest type of black holes, which have no spin or charge: $Q = J = 0$. The only parameter is the mass, M. This is called a *Schwarzschild* black hole, after Karl Schwarzschild, the Prussian astrophysicist who discovered black holes while serving on the front lines in WWI (and died a year later). (Note that he sent his solution to Einstein from the Russian front.)

- We can think of a black hole as a *point mass M*, surrounded by the light-trapping region. This point is the singularity! The *event horizon* is the boundary of the light-trapping region. Why is the horizon a sphere?
- Symmetry! A point has no preferred direction. One way to explain this is in terms of what a point "knows". As we modify spacetime (and assuming the original spacetime has no preferred direction), the only thing we can ask is "how far away is the mass?" We could ask "which direction am I pointing?", but the point will say "I don't know!" It doesn't have any information about that.
- Let's think (for just a moment) about spinning black holes. These have some axis around which they spin. What does the singularity look like?
- It can't be a point because we just argued that the horizon wouldn't have a preferred direction! The simplest guess (which turns out to be correct) is that the singularity has the shape of a ring (ringularity).
- (A follow-up question.) Why doesn't the ring singularity collapse in on itself? A qualitative answer is fine.
- Because it's spinning! Loosely, centrifugal force keeps it from collapsing.

(Say more fun stuff about the ergosphere, frame-dragging, Penrose process, etc? Also lightcones, timelike/spacelike, causality and redshift.)

Schwarzschild radius. So, let's return to the simple Schwarzschild black hole. We argued from symmetry that the horizon is a sphere. How big is the sphere? In other words, what is the point of no return? This is called the *Schwarzchild radius* R_s.

There are a couple of ways to calculate this, and you may remember Douglas Scott discussing the *escape velocity* method in the very first session last year. Instead, we'll use *dimensional analysis*. The basic idea is that physical quantities have dimensions like length L, mass M, time T, and combinations of these dimensions. (There are others but we won't need them.) We can read off dimensions from units if somebody is nice enough to give them to us.

- The speed of light is $c = 3 * 10^8$ m/s. What are the dimensions?

- This is a warm-up to remind them of rules for dimalysis, or teach them by example. We throw away numbers and use the ol' multiplicative homomorphism to the free abelian algebra generated by M, L, T (just kidding, if you say any of these things I will be upset). Ahem. We get [c] = [m/s] = [m]/[s] = L/T.
- As we (maybe) discussed above, gravity obeys an inverse square law $F = GmM/r^2$. The strength is governed by Newton's constant, $G = 6.7 * 10^{-11}$ $m^3/kg\ s^2$. What are the dimensions?
- Hopefully this is easy after the last one: [G] = L^3/MT^2.

In dimensional analysis, we have something we want to find out, like the Schwarzschild radius, we try cobbling it together out of physically relevant facts about the system. This includes relevant physical constants!

- In trying to learn the Schwarzschild radius R_s, what do you think will be relevant?
- Well, the black hole only has one parameter, the mass M. But remember that physical constants are also relevant! Since a black hole trap light with gravity, we expect c to be relevant (light) and G to be relevant (gravity).

"Cobble together" has a technical meaning: write as a product of powers. So, in our case, we guess that $R_s = G^a c^b M^d$, for some unknown powers a, b, d. The game is to figure out what the powers are by matching dimensions on both sides! The dimension on the LHS (the radius) is just length L. You turn to figure out the RHS!

- (Guide them through this.) What are the dimensions of the RHS? Express your answer in terms of the unknown powers a, b and d.
- Let's do different dimensions separately. First, length appears as L^{3a+b}. Next, time appears as $T^{-(2a+b)}$. Finally, mass appears as M^{d-a}.
- Comparing to the dimensions of the LHS, write three equations for the three unknowns.
- From length, we get 1 = 3a + b, since the LHS has length appearing as L^1 and the RHS has L^{3a+b}. Similarly, from time, we get 2a + b = 0. Finally, from mass we get d – a = 0.
- Solve the equations! What expression do you get for the Schwarzschild radius?
- We can write 1 = 3a + b = a + (2a + b) = a, so that gives us a = 1! Substituting back into the first or second equation, we find b = –2. Finally, the mass equation gives a = d = 1. Putting it all together, we have $R_s \sim GM/c^2$.
- This is only off by a factor of 2! We should probably say so, but even if they don't have it, our guess for energy released in the merger will only be wrong by an order 1 factor.

The Area Theorem. One of Stephen Hawking's most famous results is the *Area Theorem*. It states that the total area of any black hole (by which we mean the event horizon), or system of black holes, always increases. (Mention Hawking radiation.)

- Can you give an argument for the area theorem for a single "settled" black hole?

- Yes! Things can only fall into it, and never come out, by definition. Those things make the black hole heavier, and increasing M increases the Schwarzchild radius. Finally, the area $A = 4\pi R_s^2$ obviously increases with M as well, so we have a baby version of Hawking's result!
- One point: we're assuming the "settled" description holds throughout this process, i.e. it is "quasistatic". There are many ways to violate this. But Hawking's result is powerful because it applies to violent perturbations of the black hole, spinning and charged black holes, and most importantly for us, multiple colliding black holes. It's very general!
- Usually, laws of physics work the same way when you run time backwards. Can you think of any other quantity that only increases with time?
- As far as I know, *entropy* is the only such quantity. Hopefully this leads students to speculate (if they don't already know) that the area of the black hole actually measures its entropy. Many cool things to say here...

Let's now consider a more interesting case: *two* colliding black holes. The area theorem still applies, and we can use it to put an upper bound on how much energy is released in the merger! So, before the collision, we have two black holes of mass M_1 and M_2, very far apart.

- What is the total energy in the system? *Hint*. What is the most famous equation ever?
- By "very far apart" we mean we can ignore the gravitational potential of the two black holes. Then the total energy is just the sum of energy in each system. But a black hole is just a fancy point mass, and the energy is given by the GOAT equation, $E = mc^2$. I.e. the total energy is $E_{init} = (M_1+M_2)c^2$.
- What is the total area of the system?
- Let's use the correct factor of 2 in the Schwarzschild radius. Then we easily calculate that $A_{init} = 4\pi(R_1^2+R_2^2) = 16\pi(G/c^2)^2[M_1^2+M_2^2]$.

So, we now know everything about the black holes before the collision. Let's assume that after the collision, they form a spherical black hole of mass M_3. During the merger, the black holes can lose energy, most of which will be in the form of gravity waves.

- Show that, in general, $M_3 \le M_1 + M_2$. (Assume no energy is pumped into the system during the merger.)
- The initial mass $M_1+M_2 = E_{init}/c^2$. The final mass is $M_3 = E_{fin}/c^2$. If the final energy is less than the initial energy, $E_{fin} \le E_{init}$, then we immediately have $M_3 \le M_1+M_2$.
- What constraint on the mass does the area theorem give?
- For the constant $C = 16\pi(G/c^2)^2$, we have $A_{init} = C(M_1^2+M_2^2)$ and $A_{fin} = CM_3^2$. The area theorem says $A_{init} \le A_{fin}$. So we immediately get $M_1^2+M_2^2 \le M_3^2$.

We have a pair of constraints, $M_3 \le M_1 + M_2$ and $M_3^2 \ge M_1^2 + M_2^2$. The final mass is smaller than the sum but its square is bigger than the sum of squares. Let's figure out the maximum change in mass consistent with these two constraints. This will give us the maximum energy released in the merger.

- We will get the *maximum* energy lost and *minimum* energy lost by "saturating" one of these two inequalities, i.e. setting the two sides of the inequality to be *equal*, and trying to make the other one as far from equality as possible. Which should we saturate to maximise energy loss?
- If we saturate the first inequality, $M_1+M_2 = M_3$, we haven't lost energy at all! So the second inequality needs to be saturated to maximise energy loss: $M_1^2+M_2^2 = M_3^2$.
- Show that the maximum energy released is $\Delta E = [M_1 + M_2 - \sqrt{(M_1^2 + M_2^2)}]c^2$.
- Well, $\Delta E = (\Delta M)c^2$, where $\Delta M = M_1+M_2 - M_3$ is the change in mass. But if the area is unchanged, then $M_3 = \sqrt{(M_1^2+M_2^2)}$ and plugging into ΔM gives the result we want.
- (Bonus if they're keen.) Show that the maximum percentage of energy we can convert into gravitational waves in a black hole merger is ≈ 29%.
- You can do this using the quadratic formula, but there's a less tedious way using a limiting argument. Let's fix the initial mass, $M = M_1 + M_2$, and vary M_1 to maximise the energy lost. When $M_1 = 0$, we just have a single black hole and nothing happens. When $M_1 = M$, then $M_2 = 0$ and the same conclusion holds. By symmetry (or using the fact that we are maximising a quadratic), the maximum is halfway between these two roots, when $M_1 = M_2 = M/2$. This means

$$\Delta E = M_1+M_2 - \sqrt{(M_1^2+M_2^2)} = (1 - 1/\sqrt{2})M \approx 0.29M$$

- You might wonder if the maximum percentage changes when we vary the total initial mass M, but the form of the answer shows explicitly it does not.

LIGO. The first gravitational waves were detected by LIGO in 2015, from two black holes colliding 1.5 billion light years away. Let's compute how much energy was released!

- Explain why you expect the energy lost in the collision of uncharged black holes to be converted entirely to gravitational waves.
- What else can it be? The usual suspects, such as heat, light, or radiation, require matter or charge to interact. Here, there is no matter or charge, just space itself. Space can ripple due to these violent events, and this is precisely what gravitational waves are.
- The masses were $M_1 = 36M_\odot$ and $M_1 = 29M_\odot$, where $M_\odot = 2*10^{30}$ kg is the mass of our sun. Estimate the amount of energy converted into gravitational waves, expressing your answer in terms of solar masses.
- We assume the maximum was converted, and use the formula we derived earlier:

$$\Delta E = [M_1+M_2 - \sqrt{(M_1^2+M_2^2)}]c^2 = [36 + 29 - \sqrt{(36^2 + 29^2)}]M_\odot c^2 \approx 18M_\odot c^2.$$

  How did we do? Actually pretty well! The much more precise measurements of LIGO give an energy around $3M_\odot c^2$, so we are off by a factor of 6, but that is still within an order of magnitude.

- Our answer is too large by a factor of 6. Now, given the amount of handwaving we've done that's pretty good, but speculate about what we might have missed.
- We've assumed we're colliding Schwarzschild black holes. But this is real life, and these black holes are probably *spinning*, and conservation of angular momentum places additional constraints on how much energy can be released.
- How much energy actually reaches us? The collision is 1.5 billion light years away, and earth has a radius of 6000 km.
- At a distance R from the collision, all the energy is spread out over a sphere of surface area $4\pi R^2$. From very far away, earth looks like a disk of area $\pi r_{earth}{}^2$, so the energy reaching us is

$$E_{earth} = 3M_\odot c^2(\pi r_{earth}{}^2/4\pi R^2) = 6*10^{24}\text{ J}.$$

- This is still a huge amount of energy! It's about 10,000 times more energy than the whole world uses in a year. (Too large maybe?)

So, unbelievably large amounts of energy are released in black hole collisions. This raises the question: why is it so hard to detect gravitational waves? (Didn't get round to writing this part.)

– David Wakeham