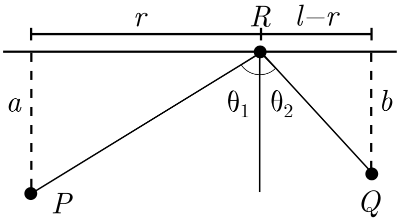
PHYC20014 Physical Systems

Classical Mechanics: Tutorial 1 Supplementary Problems

1. The Variational Biathlon. The Variational Biathlon is held at the beach and consists of two events. In Event 1, participants run as quickly as possible between two designated points on the sand, touching the shoreline en route. In Event 2, they start on the beach and race to a target point in the water using a combinaton of swimming and running. Athletes run at speed v_1 on the sand, and swim at speed v_2 in the water. You will determine the winning strategies!

(a) Argue that for trips confined entirely to one medium (sand or water), athletes should travel in straight lines.

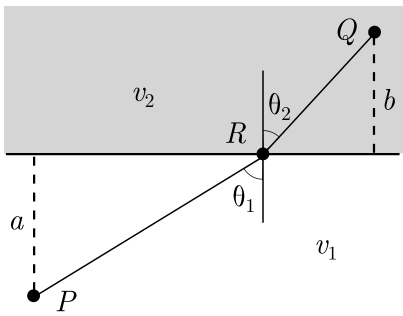


(b) In the first event (see above), athletes run from $P(0, a)$ to $Q(\ell, b)$, touching the straight shoreline ($y = 0$) in between. Let $R(r, 0)$ denote the point they touch the shoreline, with r to be determined. Show that the travel time is minimised when

$$\frac{r}{\sqrt{a^2 + r^2}} = \frac{\ell - r}{\sqrt{b^2 + (\ell - r)^2}}. \tag{1}$$

(c) Dusting off your high school trig, show that (1) implies $\theta_1 = \theta_2$ in the figure above.

(d) Solve (c) using part (a) alone.



(e) Now consider the second event (pictured above). Athletes must get from point $P(0, a)$ on the beach to $Q(\ell, -b)$ in the water, crossing from beach to ocean at $R(r, 0)$ for some r. Show that their travel time is minimised when

$$\frac{r}{v_1\sqrt{a^2 + r^2}} = \frac{\ell - r}{v_2\sqrt{b^2 + (\ell - r)^2}}, \tag{2}$$

and hence deduce that

$$v_1^{-1} \sin\theta_1 = v_2^{-1} \sin\theta_2.$$

2. POLA and mechanical energy.* The mechanical energy of the *true* path of a system can be calculated from the action in a slightly unexpected way. This insight turns out to be quite deep — in fact, it points the way to quantum mechanics — but we won't explore that here.

(a) Show from the POLA that the total mechanical energy E at the start of the trajectory satisfies

$$\frac{\partial S_{\text{true}}}{\partial t_1} = E(t_1)$$

for a general one-dimensional Lagrangian

$$L = \frac{1}{2} m\dot{x}^2 - V(x).$$

HINT. Consider separate infinitesimal changes to the starting time $t_1 \to t_1 + \delta t$ and path $x(t) \to x(t) + \delta x(t)$, where δx vanishes at t_2 but not necessarily at t_1. Combine your results using the relation between total and partial derivatives of S_{true}.

(b) Confirm the result in (a) explicitly for the action S_{true} in the tennis ball problem.

(c) What about S_{true} for a free particle? This is subtle, so be careful!

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Classical Mechanics: Tutorial 1 Supplementary Problems

Solutions

1. The Variational Biathlon.

(a) The shortest distance between two points is a straight line. Since athletes run at a uniform speed, it follows that travel time is minimised by travelling in a straight line.

(b) By (a), athletes should travel in two straight lines PR and RQ. The travel time as a function of r is therefore

$$T(r) = \frac{1}{v_1}\left[\sqrt{a^2 + r^2} + \sqrt{b^2 + (\ell - r)^2}\right].$$

Differentiating with respect to R, we obtain

$$\frac{dT}{dr} = \frac{1}{v_1}\left[\frac{r}{\sqrt{a^2 + r^2}} - \frac{\ell - r}{\sqrt{b^2 + (\ell - r)^2}}\right].$$

Setting $T' = 0$ gives us equation (1). To check this is a minimum, you can differentiate once more and show that $T'' > 0$.

(c) Looking at the diagram, the definition of sine and equation (1) imply that

$$\theta_1 = \sin^{-1}\left(\frac{r}{\sqrt{a^2 + r^2}}\right) = \sin^{-1}\left(\frac{\ell - r}{\sqrt{b^2 + (\ell - r)^2}}\right) = \theta_2.$$

(d) Ignore the ocean, pretending that sand lies on the other side of $y = 0$, and consider racing from P to the mirror image point $-Q$. Convince yourself that this is equivalent to racing from P to Q via R. From part (a), competitors should run in a straight line from P to $-Q$. Reflecting the "virtual" part of the path around $y = 0$, we see immediately that $\theta_1 = \theta_2$.

(e) This is very similar to the preceding problem. The travel time is now

$$T(r) = \frac{1}{v_1}\sqrt{a^2 + r^2} + \frac{1}{v_2}\sqrt{b^2 + (\ell - r)^2}$$

and the derivative is

$$\frac{dT}{dr} = \frac{1}{v_1}\frac{r}{\sqrt{a^2 + r^2}} - \frac{1}{v_2}\frac{\ell - r}{\sqrt{b^2 + (\ell - r)^2}}.$$

Setting $T' = 0$ gives (2), and again $T'' > 0$ ensuring this is a minimum. By similar reasoning to (c), the LHS in (2) is $v_1^{-1}\sin\theta_1$ and the RHS is $v_2^{-1}\sin\theta_2$.

2. POLA and mechanical energy.*

(a) Consider a general 1D Lagrangian $L = \frac{1}{2}m\dot{x}^2 - V(x)$ for a particle moving from x_1 at t_1 to x_2 at t_2. First, make an infinitesimal change to the path $x(t) \to x(t) + \delta x(t)$ which vanishes at the end of the motion but not necessarily at the beginning, so $\delta x_2 = 0$ but δx_1 may be nonzero. The change in the action is

$$\delta S_{\text{true}} = \int_{t_1}^{t_2} \left[L(x + \delta x, \dot{x} + \delta\dot{x}, t) - \int_{t_1}^{t_2} L(x, \dot{x}, t)\, dt \right] \equiv \int_{t_1}^{t_2} \Delta L\, dt. \tag{3}$$

We can Taylor expand L in the infinitesimal parameters $\delta x, \delta\dot{x}$:

$$\begin{aligned} \int_{t_1}^{t_2} \Delta L\, dt &= \int_{t_1}^{t_2} \left(\delta x(t) \frac{\partial L}{\partial x} + \delta\dot{x}(t) \frac{\partial L}{\partial \dot{x}} \right) dt \\ &= \left[\delta x(t) \frac{\partial L}{\partial \dot{x}} \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \delta x(t) \left(\frac{\partial L}{\partial x} - \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{x}} \right) dt \\ &= -\delta x_1 \frac{\partial L}{\partial \dot{x}}\bigg|_{t_1} + \int_{t_1}^{t_2} \delta x(t) \left(\frac{\partial L}{\partial x} - \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{x}} \right) dt. \end{aligned} \tag{4}$$

As in equation [1.6] from lectures, you can check that Newton's Second Law kills the last term in (4), since we are evaluating for the true (extremised) action S_{true}. Thus, the true action implicitly depends on the initial point x_1 via

$$\frac{\partial S_{\text{true}}}{\partial x_1} = -\frac{\partial L}{\partial \dot{x}}\bigg|_{t_1}.$$

We've also shown that there is no dependence on the initial velocity $\dot{x}(t_1) \equiv \dot{x}_1$.

Since $S_{\text{true}} = \int_{t_1}^{t_2} L(t)\, dt$, it follows that $dS_{\text{true}}/dt_1 = -L(t_1)$ from the fundamental theorem of calculus. To calculate $\partial S_{\text{true}}/\partial t_1$, we can expand the total derivative in partial derivatives and use the chain rule:

$$\begin{aligned} \frac{\partial S_{\text{true}}}{\partial t_1} &= \frac{dS_{\text{true}}}{dt_1} - \frac{\partial S_{\text{true}}}{\partial x_1}\dot{x}_1 \\ &= \dot{x}_1 \frac{\partial L}{\partial \dot{x}}\bigg|_{t_1} - L(t_1) \\ &= \frac{1}{2}m\dot{x}_1^2 + V(x_1) = E(t_1). \end{aligned}$$

This is indeed the total mechanical energy.

(b) We can set $T = t_2 - t_1$. In the tennis ball problem, for the true path we found

$$S_{\text{true}} = -\frac{T^3 mg^2}{24} = -\frac{(t_2 - t_1)^3 mg^2}{24}.$$

Noting that $\dot{x}(0) = gT/2$ and $V(0) = mgx(0) = 0$, we find

$$\frac{\partial S_{\text{true}}}{\partial t_1} = \frac{(t_1 - t_2)^2 mg^2}{8} = \frac{T^2 mg^2}{8} = \frac{1}{2}m\dot{x}(0)^2 + V(0) = E(0).$$

(c) In terms of the endpoints, $v = \Delta x/\Delta t$. The action for the free particle is therefore

$$S_{\text{true}} = \frac{1}{2}mv^2(t_2 - t_1) = \frac{1}{2}m\frac{(x_2 - x_1)^2}{t_2 - t_1}.$$

Hence,

$$\frac{\partial S_{\text{true}}}{\partial t_1} = \frac{1}{2}m\frac{(x_2 - x_1)^2}{(t_2 - t_1)^2} = \frac{1}{2}mv^2 = E(t_1).$$

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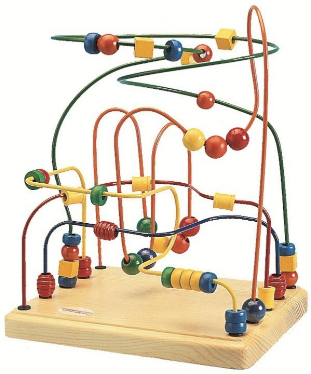
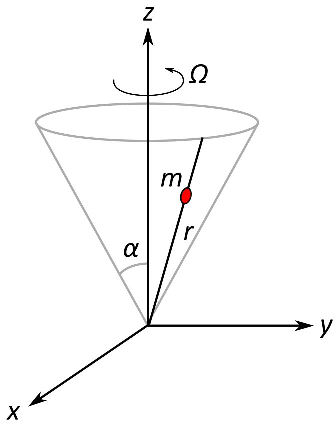
Classical Mechanics: Tutorial 2 Supplementary Problems

1. Bead on a Wire. In this problem you'll play around with a simple bead and wire system—much simpler than the systems (below left) you may have enjoyed as a child! In our simple system, a wire emanates from the origin, making an angle α with the z-axis and rotating around it with angular velocity Ω. A bead of mass m is is free to slide up and down the wire, subject only to gravity.





(a) Calculate the Lagrangian for the bead in terms of r, the distance from the bead to the origin. You should find that

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\Omega^2 \sin^2\alpha) - mgr\cos\alpha. \tag{1}$$

(b) Use Lagrange's equation to deduce the equation of motion

$$\ddot{r} = r\Omega^2 \sin^2\alpha - g\cos\alpha. \tag{2}$$

(c) Verify that (2) is solved by

$$r(t) = A\sinh(kt) + B\cosh(kt) + \frac{g\cos\alpha}{k^2}, \tag{3}$$

where $k = \Omega\sin\alpha$ and A and B are constants of integration related to the initial conditions $r(0), \dot{r}(0)$ by

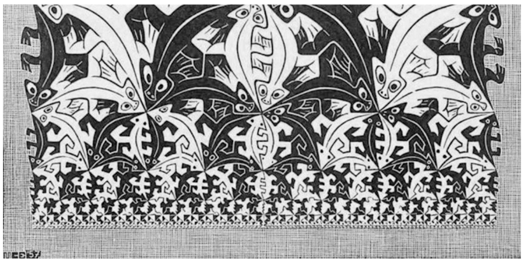
$$A = \frac{\dot{r}(0)}{k}, \quad B = r(0) - \frac{g\cos\alpha}{k^2}. \tag{4}$$

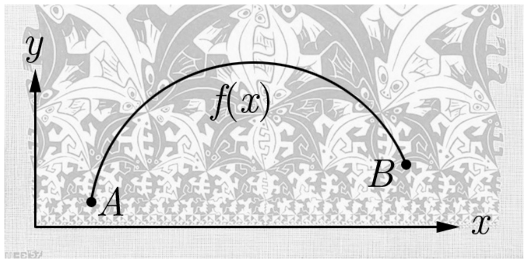
(d) At $t = 0$, the bead is launched from the origin up the wire at speed v. Assuming that $\alpha < \pi/2$, show that the bead never returns to the origin as long as

$$v \geq \left(\frac{g}{\Omega}\right)\cot\alpha.$$

Hint. You can either use (3) to evaluate $\dot{r}$ directly, or conservation arguments from the effective potential in (1).

2. Hyperbolic Lizards. One morning, you wake from troubled dreams to find yourself trapped in the Escher picture below, where distance is measured in lizards.[1] As you move towards the bottom of the picture, the lizards get smaller; equivalently, a rigid object (like a ruler) gets longer in lizard units.





2D lizard space has coordinates (x, y). At height y, the number of lizards in an interval with small coordinate displacements Δx and Δy is

$$\Delta \ell = \frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{y}.$$

The number of lizards encountered on a path $\mathcal{P}$ (parameterised by $y = f(x)$, $x \in [x_1, x_2]$) is therefore

$$\ell(\mathcal{P}) = \int_{x_1}^{x_2} \frac{\sqrt{1 + f'(x)^2}}{f(x)}\, dx \equiv \int_{x_1}^{x_2} L(f(x), f'(x), x)\, dx. \tag{5}$$

In the interests of encountering as few lizards as possible on a trip from $A(x_1, y_1)$ to $B(x_2, y_2)$, you can minimise the length of your path using Lagrange's equations.

(a) We have deliberately written (5) to remind you of the Principle of Least Action, with x playing the role of t and $f(x)$ the role of $x(t)$. Using Lagrange's equation, show that the length is minimised for a function f satisfying

$$ff'' = -[1 + (f')^2]. \tag{6}$$

(b) Check that

$$f(x) = \sqrt{R^2 - (x - k)^2} \tag{7}$$

satisfies (6). In other words, to minimise the lizards you step on, travel along arcs of circles centred on the boundary. The x-coordinate of the centre k and the radius R can always be chosen to match the endpoints A and B.[2]

[1] Apologies for Mark Van Raamsdonk for "borrowing" this joke.

[2] Well, sort of. For the case where $x_1 = x_2$, $y_1 \neq y_2$, the shortest path is a vertical line, which we can interpret as an arc on a circle with infinite radius and centre at infinity.

(c) Verify that, for a solution of (6),

$$c = L - f' \frac{\partial L}{\partial f'}$$

is a constant of motion, i.e. $c' = 0$. In terms of (7), what does c mean geometrically?

(d) *In 3D lizard space, we change to coordinates (x, y, z), with the bottom of the picture at $z = 0$. Then, for a path $\mathcal{P}(s) = (x(s), y(s), z(s))$ parameterised by a variable $s \in [s_1, s_2]$, the length in lizards is

$$\begin{aligned} \ell(\mathcal{P}) &= \int_{s_1}^{s_2} \frac{\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}}{z} \\ &= \int_{s_1}^{s_2} \frac{\sqrt{[x'(s)]^2 + [y'(s)]^2 + [z'(s)]^2}}{z(s)} \, ds \\ &\equiv \int_{s_1}^{s_2} L(x, y, z, x', y', z', s) \, ds. \end{aligned}$$

What are the ignorable coordinates and the corresponding conserved quantities? Without solving Lagrange's equations, briefly discuss what the conserved quantities imply about lizard-minimising paths, and relate this to the situation in 2D.

3. POLA and the Free Particle. In Tutorial 1, you proved that

$$\frac{\partial S_{\text{true}}}{\partial t_1} = E(t_1), \tag{8}$$

where S_{true} is the true action, t_1 is the *initial* time, and $E(t_1)$ is the total mechanical energy. But when we apply this to a free particle, with $S_{\text{true}} = mv^2(t_2 - t_1)/2$, we seem to get the wrong sign:

$$\frac{\partial S_{\text{true}}}{\partial t_1} \overset{?!}{=} -\frac{1}{2}mv^2 = -E(t_1).$$

Explain why there is actually no problem, and equation (8) is correct in this case too.

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Classical Mechanics: Tutorial 2 Supplementary Problems

Solutions

1. Bead on a Wire.

(a) In terms of r, the position of the bead is

$$x = r \sin\alpha \cos\Omega t, \quad y = r \sin\alpha \cos\Omega t \quad z = r\cos\alpha.$$

The kinetic energy is therefore

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}m(\dot{r}^2 + r^2\Omega^2 \sin^2\alpha).$$

In terms of r, the potential energy is

$$mgz = mgr\cos\alpha.$$

Hence, the Lagrangian in terms of r is

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\Omega^2\sin^2\alpha) - mgr\cos\alpha.$$

(b) Let's take some derivatives:

$$\frac{\partial L}{\partial r} = mr\Omega^2\sin^2\alpha - mg\cos\alpha, \quad \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\ddot{r}.$$

By Lagrange's equation, we equate the first and the last to find

$$\ddot{r} = r\Omega^2\sin^2\alpha - g\cos\alpha.$$

(c) Taking $r(t)$ as in (3),

$$\ddot{r}(t) = k^2\Big[A\sinh(kt) + B\cosh(kt)\Big] = r(t) - g\cos\alpha$$

as required. We note that

$$r(0) = B + \frac{g\cos\alpha}{k^2}, \quad \dot{r}(0) = kA$$

which we can invert to get (4).

(d) From equation (4),
$$A = \frac{v}{k}, \quad B = -\frac{g\cos\alpha}{k^2}.$$
Hence, the bead has position
$$r(t) = \left(\frac{v}{k}\right)\sinh(kt) - \left(\frac{g\cos\alpha}{k^2}\right)\cosh(kt) + \frac{g\cos\alpha}{k^2}$$
and velocity
$$\dot{r}(t) = v\cosh(kt) - \left(\frac{g\cos\alpha}{k}\right)\sinh(kt).$$
The velocity is always positive (ensuring the bead never returns to the origin) provided
$$v\cosh(kt) \geq \left(\frac{g\cos\alpha}{k}\right)\sinh(kt) \quad \Longrightarrow \quad \frac{vk}{g\cos\alpha} \geq \tanh(kt),$$
where we have used $\cos\alpha > 0$. Since tanh is bounded above by 1, the bead never returns to the origin as long as v satisfies
$$\frac{vk}{g\cos\alpha} \geq 1 \quad \Longrightarrow \quad v \geq \frac{g\cos\alpha}{k} = \left(\frac{g}{\Omega}\right)\cot\alpha.$$
Alternatively, from (1) we see the bead has an effective potential
$$V_{\text{eff}} \equiv \frac{1}{2}m\dot{r}^2 - L = mgr\cos\alpha - \frac{1}{2}mk^2r^2.$$
Differentiating and setting to zero, V_{eff} has a maximum at $r_{\text{max}} \equiv g\cos\alpha/k^2$, with value
$$V_{\text{max}} \equiv V_{\text{eff}}(r_{\text{max}}) = \frac{mg^2\cos^2\alpha}{2k^2}.$$
Thus, if the initial kinetic energy $T = mv^2/2 > V_{\text{max}}$, the bead flies off never to return. This is equivalent to
$$mv^2 \geq \frac{mg^2\cos^2\alpha}{k^2} \quad \Longrightarrow v \geq \frac{g\cos\alpha}{\Omega\sin\alpha} = \left(\frac{g}{\Omega}\right)\cot\alpha.$$

2. Hyperbolic Lizards.

(a) We calculate:
$$\frac{\partial L}{\partial f} = \frac{-\sqrt{1+(f')^2}}{f^2}, \quad \frac{\partial L}{\partial f'} = \frac{f'}{f\sqrt{1+(f')^2}}$$
and hence
$$\frac{d}{dx}\frac{\partial L}{\partial f'} = \frac{f''}{f\sqrt{1+(f')^2}} - \frac{(f')^2}{f^2\sqrt{1+(f')^2}} - \frac{(f')^2 f''}{f[1+(f')^2]^{3/2}}.$$
Lagrange's equation is $\partial L/\partial f = (\partial L/\partial f)'$. Multiplying by $-f^2\sqrt{1+(f')^2}$, we obtain
$$ff'' = -[1+(f')^2].$$

(b) First, note that $f'(x) = -(x-k)/f$. It follows that

$$f(x)f''(x) = -f \cdot \left(\frac{x-k}{f}\right)' = \frac{(x-k)f' - f}{f} = -\frac{f^2 + (x-k)^2}{f^2} = -[1 + (f')^2].$$

(c) We just plug and chug:

$$\begin{aligned} \frac{d}{dx}\left(L - f'\frac{\partial L}{\partial f'}\right) &= \frac{\partial L}{\partial f}f' + \frac{\partial L}{\partial f'}f'' - f''\frac{\partial L}{\partial f'} - f'\frac{d}{dx}\frac{\partial L}{\partial f'} \\ &= f'\left(\frac{\partial L}{\partial f} - \frac{d}{dx}\frac{\partial L}{\partial f'}\right). \end{aligned}$$

Since (6) is equivalent to Lagrange's equation, the last expression vanishes and $c' = 0$. From the calculations in (a) and (b),

$$c = L - f'\frac{\partial L}{\partial f'} = \frac{\sqrt{1 + (f')^2}}{f} - \frac{(f')^2}{f\sqrt{1+(f')^2}} = \frac{1}{\sqrt{f^2 + (ff')^2}} = \frac{1}{R}.$$

Thus, c^{-1} is the radius of the circle we're travelling on.

(d) The Lagrangian in 3D is

$$L(x, y, z, x', y', z', s) = \frac{\sqrt{(x')^2 + (y')^2 + (z')^2}}{z}.$$

By inspection, we see that x and y are ignorable, since $\partial L/\partial x = \partial L/\partial y = 0$. It follows from Lagrange's equations that we have conserved quantities

$$\frac{\partial L}{\partial x'} = \frac{x'}{z\sqrt{(x')^2 + (y')^2 + (z')^2}}, \quad \frac{\partial L}{\partial y'} = \frac{y'}{z\sqrt{(x')^2 + (y')^2 + (z')^2}}.$$

We square these and add them together to get another conserved quantity:

$$\left[\left(\frac{\partial L}{\partial x'}\right)^2 + \left(\frac{\partial L}{\partial x'}\right)^2\right]^{1/2} = \frac{\sqrt{(x')^2 + (y')^2}}{z\sqrt{(x')^2 + (y')^2 + (z')^2}}.$$

The numerator is the speed $|\mathcal{P}'_\perp| \equiv \sqrt{(x')^2 + (y')^2}$ as seen from above, i.e. projected onto the x–y plane. So the projected speed $|\mathcal{P}'_\perp| \propto z|\mathcal{P}'|$. As you move closer to $z = 0$, the motion must be more heavily weighted in the z direction. This is exactly what we saw in the 2D case, where we had to move along arcs of circles centred on the boundary.

3. POLA and the Free Particle. The key insight is that the velocity is a function of the endpoints, $v = \Delta x/\Delta t$. Thus, v depends on t_1! Taking this into account, the action for the free particle is

$$S_{\text{true}} = \frac{1}{2}mv^2(t_2 - t_1) = \frac{1}{2}m\frac{(x_2 - x_1)^2}{t_2 - t_1}.$$

Hence,

$$\frac{\partial S_{\text{true}}}{\partial t_1} = \frac{1}{2}m\frac{(x_2 - x_1)^2}{(t_2 - t_1)^2} = \frac{1}{2}mv^2 = E(t_1).$$

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Classical Mechanics: Tutorial 3 Supplementary Problems

1. Orbits around a black hole. Recall that for the Kepler problem (satellite of mass m orbiting a star of mass $M \gg m$), the equation of motion was

$$\frac{1}{2}\dot{r}^2 + V_{\text{eff,N}}(r) = E, \tag{1}$$

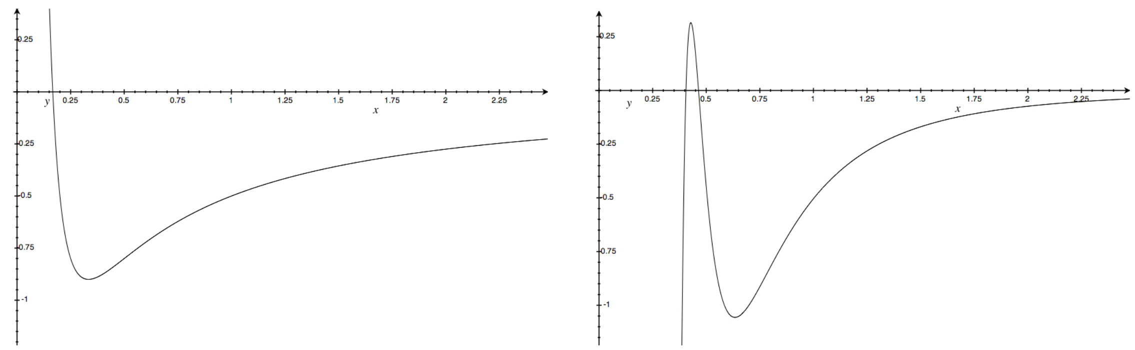
where E is the total energy of the planet and $V_{\text{eff,N}}$ the *Newtonian effective potential*:

$$V_{\text{eff,N}}(r) = \frac{J^2}{2r^2} - \frac{GM}{r}, \quad J = r^2\dot{\phi}.$$

Both J and E are constants of motion. For an orbit around a black hole, the equation of motion is the same, but the effective potential picks up a correction from general relativity:

$$V_{\text{eff}}(r) = \frac{J^2}{2r^2} - \frac{GM}{r} - \frac{GMJ^2}{c^2r^3}.$$

Here, c is the speed of light. Typical effective potentials are shown below, Newtonian on the left for comparison and black hole on the right:



(a) For the black hole, describe different possible motions of the satellite and how they depend on total energy E. Describe the qualitative difference between a Newtonian star and a black hole at small r.

(b) In terms of V_{eff}, what is the condition for a circular orbit at $r = r_0$? Show that r_0 must satisfy

$$GMr_0^2 - J^2r_0 + \frac{3GMJ^2}{c^2} = 0. \tag{2}$$

Conclude that there are no circular orbits unless $J \geq \sqrt{12}GM/c$. What happens to the graph of V_{eff} as J is lowered past $\sqrt{12}GM/c$? From now on, we assume that $J > \sqrt{12}GM/c$ so the graph above is indeed representative.

(c) Consider a small perturbation to a circular orbit,

$$r(t) = r_0 + \xi(t). \tag{3}$$

We would like to determine the equation of motion for ξ (an equation for $\ddot{\xi}$) and thereby learn the fate of the perturbation.

To find the equation, differentiate (1) with respect to time, and substitute the perturbed orbit (3) in to find an equation for $\ddot{\xi}$:

$$\ddot{\xi}(t) + V'_{\text{eff}}(r_0 + \xi) = 0.$$

Now expand $V'_{\text{eff}}(r_0 + \xi)$ as a Taylor series in ξ, and since ξ is small, throw away terms which are second order or higher. You should obtain

$$\ddot{\xi}(t) + V''_{\text{eff}}(r_0)\xi(t) = 0. \tag{4}$$

(d) Explain how solutions to (4) depend on the sign of $V''_{\text{eff}}(r_0)$. Using (2), show that the orbit is stable (the perturbation does not grow exponentially) provided

$$r_0 > \frac{6GM}{c^2}.$$

(e) Calculate the period T of the circular orbit. You should find that

$$T = 2\pi r_0 \sqrt{\frac{r_0 - 3GM/c^2}{GM}}.$$

Show that in the limit $r_0 \gg 3GM/c^2$, this matches Kepler's third law.

HINT. Use (2) and $J = r_0^2 \dot{\phi}$.

(f) For the stable perturbation in (d), what is the period of the oscillation of ξ compared to the period of the orbit? Draw the orbit. Is it closed?

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Classical Mechanics: Tutorial 3 Supplementary Problems

Solutions

1. Orbits around a black hole.

(a) For possible motions, see the graph in the lecture notes. In the Newtonian case, at small r the effective potential is dominated by the *repulsive* centrifugal barrier $\propto 1/r^2$. For the black hole, at small r the potential is dominated by an *attractive* term $\propto -1/r^3$. Given that black holes are meant to suck things in, this makes sense!

(b) A circular orbit corresponds to a local extremum of V_{eff}, so $V'_{\text{eff}}(r_0) = 0$. Doing the differentiation, we get

$$0 = V'_{\text{eff}}(r_0) = \frac{1}{r_0^4}\left[-J^2 r_0 + GMr_0^2 + \frac{3GMJ^2}{c^2}\right].$$

Since $r_0^{-4} \neq 0$, the term in brackets vanishes. Considered as a quadratic in r_0, this expression only has a solution if the discriminant is nonnegative:

$$J^4 - \frac{12(GMJ)^2}{c^2} \geq 0 \quad \Longrightarrow \quad J \geq \frac{\sqrt{12}GM}{c}.$$

If J is lowered through $\sqrt{12}GM/c$, the two extrema in the graph above coalesce when $J = \sqrt{12}GM/c$, and for $J < \sqrt{12}GM/c$ disappear.

(c) As observed earlier, a circular orbit is a local extremum of V_{eff}, with $V'_{\text{eff}}(r_0) = 0$. The Taylor expansion of $V'_{\text{eff}}(r_0 + \xi)$ is therefore

$$V'_{\text{eff}}(r_0 + \xi) = \xi V''_{\text{eff}}(r_0) + \text{higher order}.$$

Differentiating (1) with respect to time, using the chain rule and the fact that E is constant, we get

$$\dot{r}\left[\ddot{r} + V'_{\text{eff}}(r)\right] = 0.$$

But $\ddot{r} = \ddot{r}_0 + \ddot{\xi} = \ddot{\xi}$ since r_0 is constant. Dividing out $\dot{r}$ and inserting the Taylor expansion of V'_{eff},

$$\ddot{\xi} + V''_{\text{eff}}(r_0)\xi = 0.$$

(d) Let $\alpha \equiv V''_{\text{eff}}(r_0)$. For $\alpha > 0$, the solutions to (4) are *oscillatory*:

$$\xi(t) = A\cos(\sqrt{\alpha}t) + B\sin(\sqrt{\alpha}t).$$

In this case, the perturbation neither grows nor shrinks, and the orbit is stable. For $\alpha < 0$, the perturbation is *exponential*:

$$\xi(t) = Ce^{\sqrt{|\alpha|}t} + De^{-\sqrt{|\alpha|}t}.$$

Generically, $C \neq 0$ and the perturbation blows up exponentially. Our only hope for a stable orbit is oscillatory solutions. This requires $\alpha = V''_{\text{eff}}(r_0) > 0$, and hence

$$\begin{aligned} & 0 < \alpha = \frac{1}{r_0^5}\left[3J^2 r_0 - 2GMr_0^2 - \frac{12GMJ^2}{c^2}\right] \\ \implies \quad & 0 < J^2 r_0 - \frac{6GMJ^2}{c^2} - 2\left[GMr_0^2 - J^2 r_0 + \frac{3GMJ^2}{c^2}\right] \\ \implies \quad & r_0 > \frac{6GM}{c^2} \end{aligned}$$

where we have used (2) in going from the second to the third line.

(e) We can solve (2) as an equation for J, and hence $\dot{\phi}$:

$$J^2 = r_0^4 \dot{\phi}^2 = \frac{GMr_0^2}{r_0 - 3GM/c^2} \quad \implies \quad \frac{d\phi}{dt} = \frac{1}{r_0}\sqrt{\frac{GM}{r_0 - 3GM/c^2}}.$$

We can integrate $dt = (dt/d\phi)d\phi$ to get the period:

$$T = \int_0^T dt = \int_0^{2\pi} d\phi\, \frac{dt}{d\phi} = 2\pi r_0 \sqrt{\frac{r_0 - 3GM/c^2}{GM}}.$$

Kepler's third law for a circular orbit is $T_{\text{N}} = (2\pi/\sqrt{GM})r_0^{2/3}$, which clearly matches T when $r_0 \gg 3GM/c^2$.

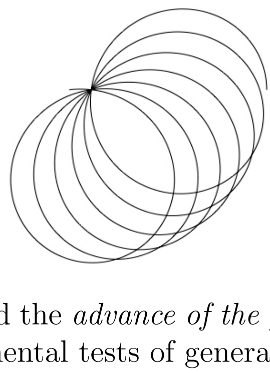
(f) From our analysis in (d), we know that the oscillation frequency of $\xi(t)$ (as distinct from the orbital frequency calculated in (e)) is $\sqrt{\alpha} = \sqrt{V''_{\text{eff}}(r_0)}$. Hence, the period is

$$T_\xi = \frac{2\pi}{\sqrt{\alpha}} = 2\pi\left[J^2 r_0^{-5}(r_0 - 6GM/c^2)\right]^{-1/2} = 2\pi r_0 \sqrt{\frac{r_0(r_0 - 3GM/c^2)}{GM(r_0 - 6GM/c^2)}}$$

where we have used the expression for J^2 from the previous question. The ratio of periods is then

$$\frac{T_\xi}{T} = \sqrt{\frac{r_0}{r_0 - 6GM/c^2}}.$$

We see that T_ξ is always *bigger* that T, so there is no way to form a closed orbit with period T. We draw the shifting orbit below:



This changing orbit shape is called the *advance of the perihelion*. Historically, this was one of the first successful experimental tests of general relativity.

PHYC20014 Physical Systems

Classical Mechanics: Tutorial 4 Supplementary Problems

1. Mechanics of Mercury.[1] The orbital mechanics of the planet Mercury are truly fascinating. The planet is in a 3:2 spin-orbit resonance, meaning that it rotates exactly three times for every two revolutions that it makes around the Sun. Similarly, the Moon is in a 1:1 spin-orbit resonance; it rotates once for every revolution it makes around the Earth, which is why we always see the same face.

Mercury is not perfectly spherical. Hence, the Sun exerts a gravitational torque upon it, keeping it locked in its 3:2 spin-orbit resonance. (Convince yourself that this torque vanishes for a spherical body.) Modern experiments[2] have measured Mercury's moment-of-inertia tensor to about three significant figures. They show that Mercury is slightly triaxial, with principal moments satisfying $I_1 < I_2 < I_3$, and

$$\frac{I_2 - I_1}{I_3} = 2.2 \times 10^{-4} \ . \tag{1}$$

See Margot, J.-L. et al. 2012, *J. Geophys. Res.*, 117, E00L09, if you want to learn more!

In this question, we calculate the potential energy U of the gravitational interaction between Mercury and the Sun, which leads to the above torque. As we learned in class, U enters into the Lagrangian for Mercury's motion.

(a) Treat the Sun as a point mass $M_\odot$ at the origin. By dividing Mercury into infinitesimal pieces, show that the gravitational potential energy is given exactly by

$$U = -GM_\odot \int \frac{d^3\mathbf{x}'\, \rho(\mathbf{x}')}{|\mathbf{x}'|} \tag{2}$$

where the integral is over the volume of the planet, ρ is the mass density, and $\mathbf{x}'$ is the displacement of an infinitesimal mass element from the Sun.

(b) Let $\mathbf{x}$ denote the displacement of the centre of mass of Mercury from the origin, and let $\mathbf{s}$ be the displacement of an infintesimal mass element from the centre of mass. Then $\mathbf{x}' = \mathbf{x} + \mathbf{s}$. Mercury is small compared to its distance from the Sun, so $|\mathbf{s}| \ll r = |\mathbf{x}|$. By Taylor expanding or otherwise, show that

$$\frac{1}{|\mathbf{x}'|} = \frac{1}{r} - \frac{\mathbf{n}\cdot\mathbf{s}}{r^2} + \frac{3(\mathbf{n}\cdot\mathbf{s})^2 - |\mathbf{s}|^2}{2r^3} \tag{3}$$

with $\mathbf{n} \equiv \mathbf{x}/r$.

[1] This question was written by Andrew Melatos.

[2] These experiments use two techniques: tracking Mercury's spin by bouncing radar echoes off the surface, and mapping its gravitational field from the trajectory of the *MESSENGER* spacecraft. *MESSENGER* (MErcury Surface, Space ENvironment, GEochemistry, and Ranging) orbited Mercury from 2011 until 2015.

(c) Substitute (3) into (2) to obtain

$$U = -\frac{GM_\odot M_{\text{Mercury}}}{r} - \frac{GM_\odot}{2r^3}\int d^3\mathbf{s}\,\rho_{\text{CM}}(\mathbf{s})\,[3(\mathbf{n}\cdot\mathbf{s})^2 - |\mathbf{s}|^2] \tag{4}$$

where $\rho_{\text{CM}}(\mathbf{s}) \equiv \rho(\mathbf{x}+\mathbf{s})$. You may assume that $\rho_{\text{CM}}(\mathbf{s}) = \rho_{\text{CM}}(-\mathbf{s})$. From now on, we use ρ to refer to the centre of mass distribution ρ_{CM}.

(d) From the general definition of the moment-of-inertia tensor I, prove that

$$\int d^3\mathbf{s}\,\rho(\mathbf{s})\,[3(\mathbf{n}\cdot\mathbf{s})^2 - |\mathbf{s}|^2] = \text{Tr}(I) - 3\mathbf{n}^{\text{T}} I\mathbf{n}\ , \tag{5}$$

where Tr denotes the trace, superscript $^{\text{T}}$ denotes the transpose, and we treat $\mathbf{n}$, $\mathbf{s}$ as column vectors. You can prove this component-wise by brute force or some other way; there are many ways to reach the answer.

(e) Mercury is almost spherical, so we have $I_1 \approx I_2 \approx I_3$. Let ψ be the angle between $\mathbf{n}$ and the principal axis $\mathbf{e}_1$. Using equations (4) and (5), deduce that

$$U = -\frac{GM_\odot M_{\text{Mercury}}}{r} - \frac{3GM_\odot(I_1 - I_2)}{2r^3}\sin^2\psi\ . \tag{6}$$

2. The phase-locked trombonist. A rhythmically-challenged trombonist tends to be out of time with the band. The band (phase θ_B) has tempo ω_B:

$$\frac{d\theta_B}{dt} = \omega_B.$$

The trombonist (phase θ) has a natural tempo $\omega \neq \omega_B$. Despite their natural inclination, the trombonist wants to stay in phase with the band, so θ evolves according to

$$\frac{d\theta}{dt} = \omega + I\sin(\theta_B - \theta), \tag{7}$$

where I measures the strength of the trombonist's response to the band. You can think of $\theta = 0,\ 2\pi,\ 4\pi,\ \ldots$ as the trombonist's beats, and $\theta_B = 0,\ 2\pi,\ 4\pi,\ \ldots$ as the band's beats. The phase difference $\psi \equiv \theta_B - \theta$ therefore satisfies

$$\frac{d\psi}{dt} = \omega_B - \omega - I\sin\psi. \tag{8}$$

(a) Explain why (7) pushes θ towards θ_B.

(b) Introducing variables $\tau = It$, $\delta = (\omega_B - \omega)/I$, show that (8) can be written

$$\frac{d\psi}{d\tau} = \delta - \sin\psi. \tag{9}$$

(c) Steady state solutions satisfy $d\psi/d\tau = 0$. This means the trombonist is *phase-locked* with the band—the tempos are the same, although their beats may be out of sync by a constant amount. Show that phase locking is only possible for

$$\omega - I \leq \omega_B \leq \omega + I.$$

When do the beats coincide?

(d) Even when phase-locked, the trombonist occasionally wanders out of phase a little. Whether they *stay* phase-locked depends on whether the solution is stable. Show that for $|\delta| < 1$ there are two solutions $\psi_1 < \psi_2$, with ψ_1 stable and ψ_2 unstable.

(e) For $|\delta| > 1$, the trombonist undergoes *phase drift*: the phase difference ψ inexorably changes with time. Show that the time it takes for ψ to change by 2π is

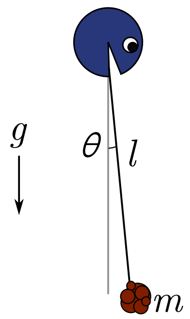
$$T = \frac{2\pi}{\sqrt{(\omega_B - \omega)^2 - I^2}}.$$

Put another way, T is the time it takes the band and trombonist to move a beat out of sync. This gives us a practical way to measure I, provided we know ω and ω_B.

HINT. You may need the integral

$$\int_0^{2\pi} \frac{d\psi}{\delta - \sin\psi} = \frac{2\pi}{\sqrt{\delta^2 - 1}}.$$

3. Spaghetti pendulum.* In an anarchic Carlton share house, spaghetti and meatballs are cooked in vast quantities each night and eaten communally. One of the housemates is a physics student, and as they slurp up a single spaghetto, they notice a meatball of mass m attached to the end.



To distract themselves from the meal, they begin speculating about the meatball's equation of motion.

(a) Write the Lagrangian for a free meatball subject to gravity. In terms of the (ℓ, θ) coordinates in the diagram, you should find

$$L = \frac{1}{2}m(\ell^2\dot{\theta}^2 + \dot{\ell}^2) + mg\ell\cos\theta.$$

Hence, derive the equations of motion

$$\ell\ddot{\theta} + 2\dot{\ell}\dot{\theta} + g\sin\theta = 0, \quad \ddot{\ell} = \ell\dot{\theta}^2 + g\cos\theta.$$

(b) Suppose the housemate slurps up a spaghetto at constant rate v. Show that for small oscillations of the meatball (i.e. small angles), the first equation of motion becomes

$$\ell\ddot{\theta} - 2v\dot{\theta} + g\theta = 0. \tag{10}$$

(c) Using the chain rule, rewrite (10) as

$$\ell\frac{d^2\theta}{d\ell^2} + 2\frac{d\theta}{d\ell} + \frac{g}{v^2}\theta = 0. \tag{11}$$

Now making the change of variables $x \equiv -2\sqrt{g\ell}/v$ and $y \equiv x\theta$, show that (11) becomes

$$x^2y'' + xy' + (x^2 - 1)y = 0 \tag{12}$$

where a dash denotes derivatives with respect to x. The differential equation (12) is solved by *Bessel functions* $J_1(x)$ and $Y_1(x)$:

$$y(x) = AJ_1(x) + BY_1(x).$$

(d) For simplicity, assume $A = 1$ and $B = 0$. Suppose the spaghetto has length L at time $t = 0$. Revert to our original variables, and express θ as a function of t. If you have a computer handy, graph θ and give a qualitative description of the dynamics of the meatball.

PHYC20014 Physical Systems

Classical Mechanics: Tutorial 3 Supplementary Problems

Solutions

1. Mechanics of Mercury.

(a) For an infinitesimal mass element $dm = d^3\mathbf{x}'\,\rho(\mathbf{x}')$ a distance $|\mathbf{x}'|$ from the origin, the gravitational potential due to the interaction with the Sun is

$$dU = -\frac{GM_\odot\,dm}{|\mathbf{x}'|}.$$

Integrating over the volume of mercury, we obtain

$$U = -GM_\odot \int \frac{d^3\mathbf{x}'\,\rho(\mathbf{x}')}{|\mathbf{x}'|}.$$

(b) First, note that

$$|\mathbf{x}'|^2 = (\mathbf{x}+\mathbf{s})\cdot(\mathbf{x}+\mathbf{s}) = r^2\left(1 + \frac{2(\mathbf{n}\cdot\mathbf{s})}{r} + \frac{|\mathbf{s}|^2}{r^2}\right).$$

Using Taylor series or the binomial expansion, for $\epsilon \ll 1$ we have

$$(1+\epsilon)^{-1/2} \simeq 1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 + \ldots$$

Combining the two, to second order in our small parameter $\mathbf{s}/r$, we have

$$\begin{aligned}
\frac{1}{|\mathbf{x}'|} &= \left[r^2\left(1 + \frac{2(\mathbf{n}\cdot\mathbf{s})}{r} + \frac{|\mathbf{s}|^2}{r^2}\right)\right]^{-1/2} \\
&\simeq \frac{1}{r}\left[1 - \frac{(\mathbf{n}\cdot\mathbf{s})}{r} - \frac{|\mathbf{s}|^2}{2r^2} + \frac{3(2\mathbf{n}\cdot\mathbf{s})^2}{8r^2}\right] \\
&= \frac{1}{r} - \frac{\mathbf{n}\cdot\mathbf{s}}{r^2} + \frac{3(\mathbf{n}\cdot\mathbf{s})^2 - |\mathbf{s}|^2}{2r^3}.
\end{aligned}$$

(c) Since r is constant, we have

$$\begin{aligned}
U &= -GM_\odot \int \frac{d^3\mathbf{x}'\,\rho(\mathbf{x}')}{|\mathbf{x}'|} \\
&= -GM_\odot \int d^3\mathbf{x}'\,\rho(\mathbf{x}')\left[\frac{1}{r} - \frac{\mathbf{n}\cdot\mathbf{s}}{r^2} + \frac{3(\mathbf{n}\cdot\mathbf{s})^2 - |\mathbf{s}|^2}{2r^3}\right] \\
&= -\frac{GM_\odot M_{\text{Mercury}}}{r} - \frac{GM_\odot}{2r^3}\int d^3\mathbf{s}\,\rho(\mathbf{s})\,[3(\mathbf{n}\cdot\mathbf{s})^2 - |\mathbf{s}|^2] + \frac{GM_\odot}{r^2}\int d^3\mathbf{s}\,\rho_{\text{CM}}(\mathbf{s})(\mathbf{n}\cdot\mathbf{s})\ .
\end{aligned}$$

On the last line, we have integrated $\rho(\mathbf{x}')$ over the volume of the planet to get the total mass M_{Mercury}. The last integral vanishes, since the integrand is odd under $\mathbf{s} \to -\mathbf{s}$:

$$\rho_{\text{CM}}(-\mathbf{s})(\mathbf{n} \cdot -\mathbf{s}) = -\rho_{\text{CM}}(\mathbf{s})(\mathbf{n} \cdot \mathbf{s}).$$

Thus, we obtain

$$U = -\frac{GM_{\odot}M_{\text{Mercury}}}{r} - \frac{GM_{\odot}}{2r^3} \int d^3\mathbf{s}\, \rho_{\text{CM}}(\mathbf{s}) \left[3(\mathbf{n} \cdot \mathbf{s})^2 - |\mathbf{s}|^2\right].$$

(d) First, we recall the matrix expression for I, using $\mathbf{s} = (x, y, z)$:

$$I = \int d^3\mathbf{s}\, \rho(\mathbf{s}) \begin{bmatrix} y^2 + z^2 & -xy & -xz \\ -yx & x^2 + z^2 & -yz \\ -zx & -zy & x^2 + y^2 \end{bmatrix}.$$

Then

$$\text{Tr}(I) = \int d^3\mathbf{s}\, \rho(\mathbf{s}) \text{Tr} \begin{bmatrix} y^2 + z^2 & -xy & -xz \\ -yx & x^2 + z^2 & -yz \\ -zx & -zy & x^2 + y^2 \end{bmatrix} = 2 \int d^3\mathbf{s}\, \rho(\mathbf{s}) |\mathbf{s}|^2.$$

Similarly, writing $\mathbf{n} = (n_x, n_y, n_z)$,

$$\begin{aligned}
\mathbf{n}^{\mathrm{T}} I \mathbf{n} &= \int d^3\mathbf{s}\, \rho(\mathbf{s}) \begin{bmatrix} n_x & n_y & n_z \end{bmatrix} \begin{bmatrix} y^2 + z^2 & -xy & -xz \\ -yx & x^2 + z^2 & -yz \\ -zx & -zy & x^2 + y^2 \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} \\
&= \int d^3\mathbf{s}\, \rho(\mathbf{s}) \begin{bmatrix} n_x & n_y & n_z \end{bmatrix} \begin{bmatrix} n_x(y^2 + z^2) - n_y xy - n_z xz \\ -n_x yx + n_y(x^2 + z^2) - n_z yz \\ -n_x zx - n_y zy + n_z(x^2 + y^2) \end{bmatrix} \\
&= \int d^3\mathbf{s}\, \rho(\mathbf{s}) \Big\{ (n_x^2 + n_y^2 + n_z^2)(x^2 + y^2 + z^2) - (n_x x + n_y y + n_z z)^2 \Big\} \\
&= \int d^3\mathbf{s}\, \rho(\mathbf{s}) \left[|\mathbf{s}|^2 - (\mathbf{n} \cdot \mathbf{s})^2\right],
\end{aligned}$$

exploiting the fact that $\mathbf{n}$ has unit norm, $n_x^2 + n_y^2 + n_z^2 = 1$. Combining these two calculations,

$$\text{Tr}(I) - 3\mathbf{n}^{\mathrm{T}} I \mathbf{n} = \int d^3\mathbf{s}\, \rho(\mathbf{s}) \left[3(\mathbf{n} \cdot \mathbf{s})^2 - |\mathbf{s}|^2\right].$$

If you're feeling brave, here is an abstract but much slicker approach using *tensor notation*. We can write the components of the matrix I as follows:

$$I_{ab} = \int d^3\mathbf{s}\, \rho(\mathbf{s}) [|\mathbf{s}|^2 \delta_{ab} - s_a s_b]$$

where δ_{ab} is the Kronecker delta, or equivalently, indexes the identity matrix. In *Einstein summation notation*, repeated indices are summed over. Einstein jokingly referred to this as his greatest mathematical discovery! In our case,

$$\mathrm{Tr}(I) = \sum_{a=1}^{3} I_{aa} \equiv I_{aa} = \int d^3\mathbf{s}\,\rho(\mathbf{s})\left[|\mathbf{s}|^2\delta_{aa} - s_a s_a\right] = \int d^3\mathbf{s}\,\rho(\mathbf{s})\cdot 2|\mathbf{s}|^2$$

since $\delta_{aa} = 3$ and $s_a s_a = |\mathbf{s}|^2$. Now we use $n_a n_a = |\mathbf{n}|^2 = 1$ to obtain

$$\begin{aligned}\mathbf{n}^{\mathrm{T}} I \mathbf{n} &= \sum_{a,b=1}^{2} n_a I_{ab} n_b \equiv n_a I_{ab} n_b \\ &= \int d^3\mathbf{s}\,\rho(\mathbf{s})\left[|\mathbf{s}|^2 n_a \delta_{ab} n_b - (n_a s_a)(n_b s_b)\right] \\ &= \int d^3\mathbf{s}\,\rho(\mathbf{s})\left[|\mathbf{s}|^2 - (\mathbf{n}\cdot\mathbf{s})^2\right]\end{aligned}$$

since $n_a s_a = \mathbf{n}\cdot\mathbf{s}$. Much nicer!

(e) Using principal axes diagonalises the moment of inertia tensor: $I = \mathrm{diag}(I_1, I_2, I_3)$. The normal vector $\mathbf{n}$ makes an angle ψ with the $\mathbf{e}_1$-axis, and also subtends a polar angle ϕ from the $\mathbf{e}_2$-axis in the $\mathbf{e}_{2,3}$-plane. So, in principal coordinates, we can write $\mathbf{n} = (\cos\psi, \sin\psi\cos\phi, \sin\psi\sin\phi)$, and

$$\begin{aligned}\mathrm{Tr}(I) - 3\mathbf{n}^{\mathrm{T}} I \mathbf{n} &= I_1(1 - 3\cos^2\psi) + I_2(1 - 3\sin^2\psi\cos^2\psi) + I_3(1 - 3\sin^2\sin\cos^2\psi) \\ &= 3(I_1 - I_2\cos^2\phi - I_3\sin^2\phi)\sin^2\psi - 2I_1 + I_2 + I_3 \\ &\approx 3(I_1 - I_2)\sin^2\psi\end{aligned}$$

where we have used $I_1 \approx I_2 \approx I_3$ on the last line. Subbing into (4) and (5),

$$U = -\frac{GM_\odot M_{\mathrm{Mercury}}}{r} - \frac{3GM_\odot(I_1 - I_2)}{2r^3}\sin^2\psi\ .$$

2. The phase-locked trombonist.

(a) Assume $I > 0$. If the trombonist is lagging, with $\theta < \theta_B$, the sine term will speed $\dot\theta$ up; similarly, if the trombonist is too fast, the the sine term slows them down.

(b) From (8), we have

$$\frac{d\psi}{d\tau} = \frac{d\psi}{I dt} = \frac{\omega_B - \omega}{I} - \sin\psi = \delta - \sin\psi.$$

(c) Setting the LHS of (9) to zero, we get

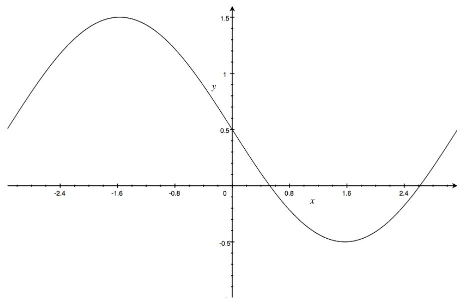
$$\delta = \sin\psi.$$

This is only possible for $|\delta| \le 1$, or equivalently,

$$\omega - I \le \omega_B \le \omega + I.$$

Beats coincide when $\psi = 0$, or $\delta = 0$. This means beats never coincide for steady state solutions unless $\omega_B = \omega$!

(d) Let's draw $d\psi/d\tau$ vs ψ for $\delta = 0.5$:



Steady state solutions correspond to the zeros near $\psi_1 \approx 0.5$ and $\psi_2 \approx 3$; The solution ψ_1 is stable since increasing ψ a little make $d\psi/d\tau$ *negative*, tending to restore it to ψ_1, and likewise for decreasing ψ. For the same reason, ψ_2 is unstable; push it a little and it moves further away. Changing δ (but keeping $|\delta| < 1$) only changes the position of the zeros, but doesn't affect their stability.

(e) We use the same trick from ordinary mechanics problems:

$$T = \int_{\psi=0}^{\psi=2\pi} dt = \int_0^{2\pi} \frac{dt}{d\psi}\, d\psi = \int_0^{2\pi} \frac{d\psi}{\delta - \sin\theta} = \frac{2\pi}{\sqrt{\delta^2 - 1}}.$$

Notice that as $\delta \to \pm 1$, the period goes off to infinity: if the trombonist is phase-locked, they never skip a beat!

3. Spaghetti pendulum.*

(a) Converting from Cartesian to polar coordinates,

$$x = \ell \cos\theta, \quad \dot{x} = -\ell\dot{\theta}\sin\theta + \dot{\ell}\cos\theta$$
$$y = \ell \sin\theta, \quad \dot{y} = \ell\dot{\theta}\cos\theta + \dot{\ell}\sin\theta.$$

Hence, a little algebra shows that

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m(\ell^2\dot{\theta}^2 + \dot{\ell}^2).$$

Gravity pulls in the positive x direction, so

$$V = -mgx = -mg\ell\cos\theta.$$

Hence,

$$L = T - V = \frac{1}{2}m(\ell^2\dot{\theta}^2 + \dot{\ell}^2) + mg\ell\cos\theta.$$

For Lagrange's equations, we need

$$\frac{\partial L}{\partial \theta} = -mg\ell\sin\theta, \quad \frac{\partial L}{\partial \dot{\theta}} = m\ell^2\dot{\theta}, \quad \frac{\partial L}{\partial \ell} = m\ell\dot{\theta}^2 + mg\cos\theta, \quad \frac{\partial L}{\partial \dot{\ell}} = m\dot{r}.$$

From Lagrange's equations, we therefore obtain

$$\ell\ddot{\theta} + 2\dot{\ell} + g\sin\theta = 0, \quad \ddot{\ell} = \ell\dot{\theta}^2 + g\cos\theta.$$

(b) If the housemate slurps spaghetti at a constant rate v, the length of pendulum changes at rate $\dot{\ell} = -v$. For small oscillations of the meatball, we use the approximation $\sin\theta \approx \theta$. Hence, the equation of motion for θ becomes

$$\ell\ddot{\theta} - 2v + g\theta = 0.$$

(c) We note from the chain rule that

$$\dot{\theta} = \frac{d\theta}{d\ell}\frac{d\ell}{dt} = -v\frac{d\theta}{d\ell}, \quad \dot{\theta} = v^2\frac{d^2\theta}{d\ell^2}.$$

Substituting into (10), we get

$$\ell\frac{d^2\theta}{d\ell^2} + 2\frac{d\theta}{d\ell} + \frac{g}{v^2}\theta = 0.$$

First, change variables to $x \equiv -2\sqrt{g\ell}/v$. After some algebra, we get

$$x\theta'' + 3\theta' + x\theta = 0,$$

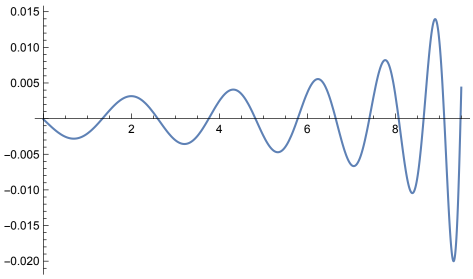
where dashes denote x derivatives. Now make the change of variable $y = x\theta$. After some more algebra, we get

$$x^2y'' + xy' + (x^2 - 1)y = 0.$$

(d) Let's go backwards:

$$\begin{aligned} y(x) = J_1(x) \quad &\implies \quad \theta(x) = \frac{1}{x}J_1(x) \\ &\implies \quad \theta(t) = -\frac{v}{2\sqrt{g(L-vt)}}J_1\left[-\frac{2\sqrt{g(L-vt)}}{v}\right] \end{aligned}$$

where on the last line, we use $\ell(t) = L - vt$. We plot this for a slurping rate of 5 cm/s, initial spaghetto length $L = 50$ cm, and normal gravity $g = 10$ m/s^2:



As you might expect from everyday experience, as the spaghetto gets shorter, the meatball oscillations become faster and wider.

PHYC20014 Physical Systems

Classical Mechanics: Tutorial 5 Problems

1. Canonical transformations. In Lagrangian mechanics, we can choose coordinates at will; the Principle of Least Action (or equivalently, Lagrange's Equation) does not depend on how we label configuration space. Hamiltonian mechanics is a little bit different: Hamilton's equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{1}$$

are not invariant under any old change of phase space coordinates $(q_i, p_i) \to (Q_i(q,p), P_i(q,p))$. A coordinate transformation preserving (1) is called a *canonical transformation.*

(a) In lectures, we derived the Hamiltonian for a 2D simple harmonic oscillator. In 1D, the Hamiltonian is even simpler:

$$H(q,p) = \frac{p^2}{2m} + \frac{1}{2}kq^2 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2. \tag{2}$$

Rotate coordinates in phase space by an angle θ:

$$Q(q,p) = q\cos\theta - p\sin\theta, \quad P(q,p) = q\sin\theta + p\cos\theta.$$

Show that Q and P evolve according to

$$\begin{aligned}
\dot{Q} &= P\left[\frac{1}{m}\cos^2\theta + m\omega^2\sin^2\theta\right] + Q\cos\theta\sin\theta\left[m\omega^2 - \frac{1}{m}\right] \\
\dot{P} &= -Q\left[\frac{1}{m}\sin^2\theta + m\omega^2\cos^2\theta\right] + P\cos\theta\sin\theta\left[\frac{1}{m} - m\omega^2\right].
\end{aligned}$$

Write down the Hamiltonian $H(Q,P)$ in the new coordinates. When is the transformation canonical?

(b) Suppose the transformation $(q_i, p_i) \to (Q_i, P_i)$ is canonical. Argue that the Poisson bracket is automatically preserved:

$$\{f,g\}_{(q,p)} \equiv \frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j} - \frac{\partial g}{\partial q_j}\frac{\partial f}{\partial p_j} = \frac{\partial f}{\partial Q_j}\frac{\partial g}{\partial P_j} - \frac{\partial g}{\partial Q_j}\frac{\partial f}{\partial P_j} \equiv \{f,g\}_{(Q,P)}.$$

Also derive the converse: a transformation preserving the Poisson bracket is canonical.

(c) Consider an infinitesimal change of coordinates,

$$\begin{aligned}
q_i \to Q_i &= q_i + \epsilon A_i(q,p) \\
p_i \to P_i &= p_i + \epsilon B_i(q,p).
\end{aligned}$$

Show this transformation is canonical if there is some smooth function $G(q,p)$ satisfying

$$A_i = \frac{\partial G}{\partial p_i}, \quad B_i = -\frac{\partial G}{\partial q_i}.$$

This looks a lot like Hamilton's equations. In fact, setting $A_i = \dot{q}_i$, $B_i = \dot{p}_i$, and $G = H$, we see that evolving the system an infinitesimal time, $t \to t + \epsilon$, is canonical.

2. Action-angle variables. We now see how canonical transformations can make life simpler in Hamiltonian mechanics. For the 1D SHO, make the transformation $(q, p) \to (\theta, I)$ given by

$$q = \sqrt{\frac{2I}{m\omega}} \sin\theta, \quad p = \sqrt{2Im\omega} \cos\theta. \tag{3}$$

The new coordinates are called *action-angle variables.*

(a) Demonstrate that $(q, p) \to (\theta, I)$ is canonical using the Poisson bracket.

(b) Derive the Hamiltonian in the new variables. You should find

$$H(\theta, I) = \omega I.$$

(c) What are Hamilton's equations? Draw the corresponding trajectories in phase space.

(d) Consider a 1D Hamiltonian

$$H(q, p) = \frac{p^2}{2m} + V(q)$$

exhibiting periodic motion. The corresponding action variable I is the area of phase space enclosed in a single orbit divided by 2π:

$$I = \frac{A}{2\pi}.$$

Check that this agrees with (3).

3. Delays and social media. Consider a toy model of the popularity P of a topic on social media,

$$\dot{P}(t) = \dot{P}(t - T) + A[P(t) - P(t - T)]. \tag{4}$$

The first term on the right measures the response to trending, while the last two (governed by the "activity" A) correspond to random surfing onto new topics or away from old ones. Finally, $T > 0$ is the characteristic time lag for users to respond to new items.

(a) Trial an exponential $P(t) = e^{\lambda t}$ in (4). Show that λ must satisfy the equation

$$(\lambda - A)(1 - e^{-\lambda T}) = 0. \tag{5}$$

(b) Show that (5) has solutions $\lambda = A$ and $\lambda = im\omega$ for $\omega \equiv 2\pi/T$, $m \in \mathbb{Z}$.

(c) For $A > 0$, the solution $\lambda = A$ corresponds to runaway growth, i.e. something going *viral.* In terms of (4), explain qualitatively how this happens.

(d) The solutions $\lambda = im\omega$ correspond to periodic fluctuations in popularity. In terms of (4), explain qualitatively how oscillations arise. Evidently, for this model, trending alone cannot generate viral success.

PHYC20014 Physical Systems

Classical Mechanics: Tutorial 5 Problems

Solutions

1. Canonical transformations.

(a) Using Hamilton's equations (1) and the Hamiltonian (2) for (p, q), the rotated variables (Q, P) evolve according to

$$\dot{Q} = \dot{q}\cos\theta - \dot{p}\sin\theta = \frac{p}{m}\cos\theta + m\omega^2 q\sin\theta,$$
$$\dot{P} = \dot{q}\sin\theta + \dot{p}\cos\theta = \frac{p}{m}\sin\theta - m\omega^2 q\cos\theta.$$

We can rotate by $-\theta$ to express the (q, p) in terms of (Q, P):

$$q = Q\cos\theta + P\sin\theta, \quad p = -Q\sin\theta + P\cos\theta.$$

After a little algebra, we get

$$\dot{Q} = P\left[\frac{1}{m}\cos^2\theta + m\omega^2\sin^2\theta\right] + Q\cos\theta\sin\theta\left[m\omega^2 - \frac{1}{m}\right]$$
$$\dot{P} = -Q\left[\frac{1}{m}\sin^2\theta + m\omega^2\cos^2\theta\right] + P\cos\theta\sin\theta\left[\frac{1}{m} - m\omega^2\right].$$

Now write the Hamiltonian in the new variables:

$$\begin{aligned} H(Q, P) &= \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 \\ &= \frac{1}{2m}\Big[-Q\sin\theta + P\cos\theta\Big]^2 + \frac{1}{2}m\omega^2\Big[Q\cos\theta + P\sin\theta\Big]^2 \\ &= Q^2\left[\frac{1}{2m}\sin^2\theta + \frac{1}{2}m\omega^2\cos^2\theta\right] + P^2\left[\frac{1}{2m}\cos^2\theta + \frac{1}{2}m\omega^2\sin^2\theta\right] \\ &\qquad + PQ\cos\theta\sin\theta\left[m\omega^2 - \frac{1}{m}\right]. \end{aligned}$$

This is a horrible mess! We will see a much nicer change of coordinates in Problem 2. Finally, let's see if the system obeys (1):

$$\frac{\partial H}{\partial P} = P\left[\frac{1}{m}\cos^2\theta + m\omega^2\sin^2\theta\right] + Q\cos\theta\sin\theta\left[m\omega^2 - \frac{1}{m}\right] = \dot{Q}$$
$$\frac{\partial H}{\partial Q} = Q\left[\frac{1}{m}\sin^2\theta + m\omega^2\sin^2\theta\right] + P\cos\theta\sin\theta\left[m\omega^2 - \frac{1}{m}\right] = -\dot{P}.$$

We see that the transformation is canonical for all θ. In fact, rotations are canonical for *any* 2D phase space. For higher-dimensional phase space, this no longer holds.

(b) By definition, a canonical transformation satisfies (1). Let's unpack this a bit using the chain rule and Hamilton's equations for q and p:

$$\begin{aligned}
\frac{\partial H}{\partial P_i} = \dot{Q}_i &= \frac{\partial Q_i}{\partial q_j}\dot{q}_j + \frac{\partial Q_i}{\partial p_j}\dot{p}_j \\
&= \frac{\partial Q_i}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial Q_i}{\partial p_j}\frac{\partial H}{\partial q_j} \\
&= \frac{\partial Q_i}{\partial q_j}\left(\frac{\partial H}{\partial Q_k}\frac{\partial Q_k}{\partial p_j} + \frac{\partial H}{\partial P_k}\frac{\partial P_k}{\partial p_j}\right) - \frac{\partial Q_i}{\partial p_j}\left(\frac{\partial H}{\partial Q_k}\frac{\partial Q_k}{\partial q_j} + \frac{\partial H}{\partial P_k}\frac{\partial P_k}{\partial q_j}\right). \qquad (6)
\end{aligned}$$

Comparing the first and last terms, we can immediately read off that

$$\frac{\partial Q_i}{\partial q_j}\frac{\partial P_k}{\partial p_j} - \frac{\partial Q_i}{\partial p_j}\frac{\partial P_k}{\partial q_j} = \{Q_i, P_k\}_{(q,p)} = \delta_{ik} \qquad (7)$$

$$\frac{\partial Q_i}{\partial q_j}\frac{\partial Q_k}{\partial p_j} - \frac{\partial Q_i}{\partial p_j}\frac{\partial Q_k}{\partial q_j} = \{Q_i, Q_k\}_{(q,p)} = 0. \qquad (8)$$

Similarly, the equation for $\dot{P}_i$ tells us that

$$\{P_i, P_k\}_{(q,p)} = 0. \qquad (9)$$

With (8) and (9), we can simplify the first term in the Poisson bracket $\{f, g\}_{(q,p)}$:

$$\begin{aligned}
\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j} &= \left(\frac{\partial f}{\partial Q_k}\frac{\partial Q_k}{\partial q_j} + \frac{\partial f}{\partial P_k}\frac{\partial P_k}{\partial q_j}\right)\left(\frac{\partial g}{\partial Q_\ell}\frac{\partial Q_\ell}{\partial p_j} + \frac{\partial g}{\partial P_\ell}\frac{\partial P_\ell}{\partial p_j}\right) \\
&= \frac{\partial f}{\partial Q_k}\frac{\partial Q_k}{\partial q_j}\frac{\partial g}{\partial Q_\ell}\frac{\partial Q_\ell}{\partial p_j} + \frac{\partial f}{\partial Q_k}\frac{\partial Q_k}{\partial q_j}\frac{\partial g}{\partial P_\ell}\frac{\partial P_\ell}{\partial p_j} \\
&\quad + \frac{\partial f}{\partial P_k}\frac{\partial P_k}{\partial q_j}\frac{\partial g}{\partial Q_\ell}\frac{\partial Q_\ell}{\partial p_j} + \frac{\partial f}{\partial P_k}\frac{\partial P_k}{\partial q_j}\frac{\partial g}{\partial P_\ell}\frac{\partial P_\ell}{\partial p_j}.
\end{aligned}$$

Thus, swapping dummy indices $l \leftrightarrow k$ where convenient and using (7)–(9),

$$\begin{aligned}
\{f, g\}_{(q,p)} &= \{f, g\}_{(Q,,P)}\{Q_k, P_\ell\}_{(q,p)} + \frac{\partial f}{\partial Q_k}\frac{\partial g}{\partial Q_\ell}\{Q_k, Q_\ell\}_{(q,p)} + \frac{\partial f}{\partial P_k}\frac{\partial g}{\partial P_\ell}\{P_k, P_\ell\}_{(q,p)} \\
&= \{f, g\}_{(Q,P)}.
\end{aligned}$$

Done! Going backwards, suppose that the Poisson bracket is preserved. Then (7), (8) and (9) automatically follow. But it is not hard to see that these are *equivalent* to the identity in (6) and the corresponding result for $\dot{P}_i$. In turn, these imply Hamilton's equations in the (Q, P) coordinates.

You may be comforted to know that there are slicker techniques for doing these calculations, namely the *symplectic approach*. See §9.4 of Goldstein, Poole and Safko for more detail.

(c) We need Poisson brackets (7)–(9) to be conserved. Start with $\{Q_i, P_k\}_{(q,p)} = \delta_{ik}$, discarding terms of order ϵ^2:

$$
\begin{aligned}
\{Q_i, P_k\}_{(q,p)} &= \frac{\partial Q_i}{\partial q_j}\frac{\partial P_k}{\partial p_j} - \frac{\partial P_k}{\partial q_j}\frac{\partial Q_i}{\partial p_j} \\
&= \left(\delta_{ij} + \epsilon\frac{\partial A_i}{\partial q_j}\right)\left(\delta_{kj} + \epsilon\frac{\partial B_k}{\partial p_j}\right) - \epsilon^2\frac{\partial A_i}{\partial p_j}\frac{\partial B_k}{\partial q_j} \\
&\simeq \delta_{ik} + \epsilon\left(\frac{\partial A_i}{\partial q_k} + \frac{\partial B_k}{\partial p_i}\right). \qquad (10)
\end{aligned}
$$

We need this last term to vanish if the transformation is to be canonical. Assume there is some smooth function $G(q,p)$ satisfying

$$
A_i = \frac{\partial G}{\partial p_i}, \quad B_i = -\frac{\partial G}{\partial q_i}.
$$

Since partial derivatives of G commute, we immediately obtain

$$
\frac{\partial A_i}{\partial q_k} = \frac{\partial^2 G}{\partial q_k \partial p_i} = \frac{\partial^2 G}{\partial p_i \partial q_k} = -\frac{\partial B_k}{\partial p_i}.
$$

Thus, the last term in (10) vanishes and the transformation is canonical. You can easily check that the remaining Poisson brackets are satisfied.

2. Action-angle variables.

(a) As discussed in Problem 1, to show a transformation is canonical, we only need to check the Poisson brackets (7)–(9). In fact, since we have defined p and q in terms of I and θ, it is easier (but equivalent) to verify the usual q,p Poisson brackets in the (θ, I) coordinates:

$$
\begin{aligned}
\{q,p\}_{\theta,I} &= \frac{\partial q}{\partial \theta}\frac{\partial p}{\partial I} - \frac{\partial q}{\partial I}\frac{\partial p}{\partial \theta} \\
&= \sqrt{\frac{2I}{m\omega}}\cos\theta \cdot \sqrt{\frac{m\omega}{2I}}\cos\theta + \sqrt{\frac{1}{2Im\omega}}\sin\theta \cdot \sqrt{2Im\omega}\sin\theta \\
&= \cos^2\theta + \sin^2\theta = 1.
\end{aligned}
$$

The remaining brackets $\{q,q\}_{\theta,I} = \{p,p\}_{\theta,I} = 0$ follow immediately by antisymmetry. So the action-angle variables are indeed canonical.

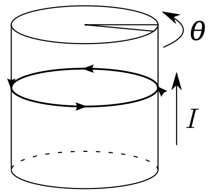
(b) We simply substitute (3) into (2):

$$
H(\theta, I) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 = \frac{2Im\omega}{2m}\cos^2\theta + \frac{2Im\omega^2}{2m\omega}\sin^2\theta = \omega I.
$$

(c) Hamilton's equations (1) become

$$\dot{\theta} = \frac{\partial H}{\partial I} = \omega, \quad \dot{I} = -\frac{\partial H}{\partial \theta} = 0.$$

In other words, I is conserved on phase space trajectories, while θ rotates with angular frequency ω. We could draw these on the phase plane, but the dynamics is better represented on a *cylinder*:



(d) Since energy is conserved, $H = E$ on phase space trajectories. So, for the 1D SHO, the trajectory is an ellipse:

$$1 = \frac{p^2}{2mE} + \frac{q^2}{2(m\omega^2)^{-1}E} \equiv \frac{p^2}{a^2} + \frac{q^2}{b^2}.$$

The area of the ellipse is therefore

$$A = \pi ab = \pi\sqrt{2mE \cdot 2(m\omega^2)^{-1}E} = \frac{2\pi H}{\omega} = 2\pi I.$$

3. Delays and viral marketing.

(a) Substituting $P(t) = e^{\lambda t}$ into (4), we obtain

$$\lambda e^{\lambda t} = \lambda e^{-\lambda T} e^{\lambda t} + Ae^{\lambda t}(1 - e^{-\lambda T}).$$

Dividing by $e^{\lambda t}$ and rearranging, we obtain (5).

(b) We need to make either factor of (5) vanish. Clearly, the first factor vanishes for $\lambda = A$. The second factor vanishes when $e^{-\lambda T} = 1$, or

$$\lambda T = 2\pi m i, \quad m \in \mathbb{Z}.$$

Equivalently, $\lambda = im\omega$, where $\omega \equiv 2\pi/T$.

(c) For runaway growth, trending and random browsing *away* from the topic cancel out, leaving the random browsing *onto* the topic:

$$\dot{P}(t) = AP(t).$$

This creates exponential, "Zahir"-type viral success!

(d) In this case, the random browsing onto and away from the topic cancel, leaving the response to trending:

$$\dot{P}(t) = \dot{P}(t - T).$$

This is solved by periodic functions whose period T_P fits neatly into T, i.e. $T = nT_p$ for some positive integer n.