UBC Physics Circle: Problems & Solutions

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Abstract

Questions I've contributed to the UBC Physics Circle (2018–19). None of them require calculus, but they do assume some problem-solving maturity and a strong background in high school physics and maths. All material here is original, though I draw on a variety of inspirations and sources, and list specific references where possible. Feel free to use problems, but please cite the author. If you have any corrections, please contact me at `<david.a.wakeham@gmail.com>` .

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1 Motion

1.1 Gone fishin'

After a day hard at work on kinematics, Emmy decides to take a break from physics and go fishing in nearby Lake Lagrange. But there is no escape! As she prepares to cast her lure, she realises she has an interesting ballistics problem on her hands.

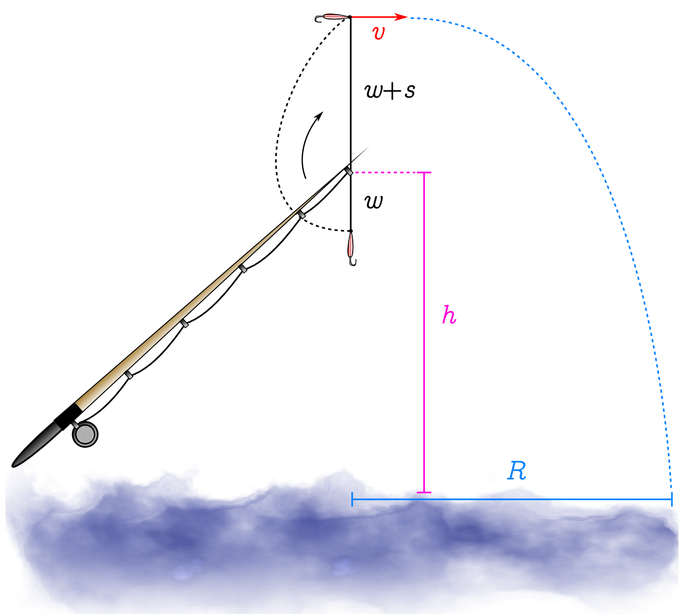


Figure 1: Emmy's unconventional method for casting a lure.

1. The top of her rod is a distance h above the water, and the lure (mass m) hangs on a length of fishing wire w. To cast, Emmy will swing the lure $180°$ around the end of the rod and release at the highest point, where the velocity has no vertical component. Assuming she can impart angular momentum $L = mvw$ to the lure, calculate the range R in terms of h, w, L and m. You can ignore the effect of gravity during the swinging phase.

2. As Emmy swings through, she can introduce some additional slack s into the wire. Assuming conservation of angular momentum, this will slow the lure but raise the release point. Find the range R in terms of the parameters w, s, h, L, and determine the amount of slack s that maximises the casting distance. Again, ignore the effect of gravity.

   Hint. Try maximising the *square* of the range.

3. Now include gravity in the swinging phase, and calculate the range as a function of s. Determine the optimum s, and find a condition on h, w, L, g, m which ensures $s > 0$.

   Hint. Complete the square in R^2.

Solution

1. Since the lure is released with no vertical velocity, the time it takes to hit the water is

$$h + w = \frac{1}{2}gt^2 \quad \Longrightarrow \quad t = \sqrt{\frac{2h}{g}}.$$

The "muzzle" velocity is $v = L/mw$, so the range r of the lure is

$$R = \frac{L}{mw}\sqrt{\frac{2(h+w)}{g}}.$$

2. Our previous answer for range is simply modified by making the replacement $w \to w + s$, but keeping the angular momentum L fixed:

$$R = \frac{L}{m(w+s)}\sqrt{\frac{2(h+w+s)}{g}}.$$

We would like to maximise this distance. We can ignore the constants L, m and $g/2$, write $x = w + s$, and focus on maximising

$$f(x) = \frac{\sqrt{h+x}}{x}.$$

Since this is positive, we can maximise this just as well by maximising its *square* as the hint suggests:

$$F(x) = f^2(x) = \frac{h+x}{x^2}.$$

It's not hard to show that this is a *decreasing* function, so that the best strategy is for Emmy to introduce no slack at all. Let's check that this is true, assuming $0 < x < z$ and trying to show that $F(x) > F(y)$, or even better, $F(x) - F(z) > 0$. We have

$$\begin{aligned} F(x) - F(z) &= \frac{h+x}{x^2} - \frac{h+z}{z^2} \\ &= \frac{(h+x)z^2 - (h+z)x^2}{x^2z^2} \\ &= \frac{h(z^2 - x^2) + xz(z-x)}{x^2z^2}. \end{aligned}$$

Since $z > x$, we have $z^2 > x^2$, so the numerator is positive. The denominator is also positive, which means that the whole expression is positive! So the maximum range occurs for $s = 0$.

3. If Emmy adds slack s during the swing, then the lure will undergo a change in height $\Delta y = 2w + s$. This causes the lure to gain gravitational potential energy

$$\Delta U = mg\Delta y = mg(2w+s),$$

leading to a reduced release velocity v':

$$\Delta K = \frac{1}{2}m[(v')^2 - v^2] = -\Delta U \quad \Longrightarrow \quad v' = \sqrt{v^2 - 2g(2w+s)}.$$

Plugging in $v = L/mw$, the range is now

$$R = \sqrt{\frac{2(h+w+s)}{g}\left[\frac{L^2}{m^2w^2} - 2g(2w+s)\right]}.$$

The question now is how to optimise this horrible looking expression! Once again, we can square R, throw away some constants which sit out front, and maximise the very simple function

$$F(s) = (A+s)(B-s),$$

where

$$A = h + w, \quad B = \frac{L^2}{2gm^2w^2} - 2w.$$

By completing the square, we can write

$$F(s) = -\left(s - \tfrac{1}{2}(A-B)\right)^2 + \frac{1}{4}(A-B)^2.$$

Only the first part is relevant to figuring out the optimal s. The function $F(s)$ will be maximised for

$$s = \frac{1}{2}(A-B) = h + 3w - \frac{L^2}{2gm^2w^2}.$$

Of course, for this to be positive, we require $A > B$, or equivalently

$$h + 3w > \frac{L^2}{2gm^2w^2}.$$

1.2 Snowballing

Recall the formula for *impulse*, stating that a force applied over time will lead to a change in momentum:

$$F_{\text{avg}}\Delta t = \Delta p.$$

The force can change, with F_{avg} denoting the *average* over the interval. We can approximate the average force as the arithmetic average of the applied force at the start and the end of the interval:

$$F_{\text{avg}} \approx \frac{F_{\text{start}} + F_{\text{end}}}{2}.$$

The impulse formula works for an object with *changing mass*. In fact, we can view it as the

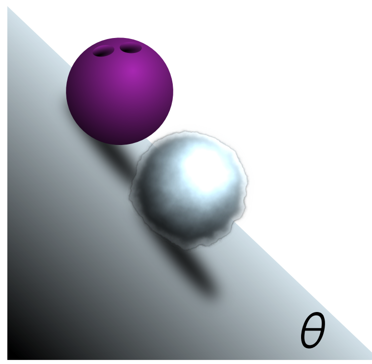


Figure 2: A snowball and a bowling ball racing down a mountainside.

most general statement of Newton's second law! For an object with constant mass m, we can write

$$F = ma = m\frac{\Delta v}{\Delta t} = \frac{\Delta(mv)}{\Delta t} = \frac{\Delta p}{\Delta t}.$$

The rightmost expression is precisely the impulse formula, and works perfectly well for an object with changing mass.

As an example, consider a snowball of initial mass m_0 and radius r_0 sitting on top of a mountain. A northerly begins to blow, dislodging the snowball and causing it to roll down the mountainside and accumulate snow. We will assume the snowball has uniform density ρ and the slope is constant.

1. First, suppose that the snowball is frozen solid with constant mass. If the mountain slopes at an angle θ to the horizontal, what is the snowball's linear accleration a_0?

   Hint. The moment of inertia of the snowball is $I = (2/5)m_0 r_0^2$.

2. Now suppose that in a short time increment Δt, the snowball picks up a mass Δm. Show that the acceleration over the interval $a = \Delta v/\Delta t$ is related to the acceleration a_0 of the constant mass snowball by

$$(m + \Delta m)a = \left(m + \frac{1}{2}\Delta m\right) a_0 - v\frac{\Delta m}{\Delta t}.$$

3. Taking the limit of a very small time interval Δt, argue that

$$a = a_0 - \frac{v}{m}\frac{\Delta m}{\Delta t}.$$

4. A bowling ball (constant mass and density) races the snowball down the mountainside. If the snowball is gathering snow, which arrives at the bottom of the slope first? If the sun comes out, and the snowball melts as it travels down the slope instead of getting heavier, what happens then?

5. Suppose that the rate of accumulation $\Delta m/\Delta t > 0$ is proportional to the *surface area* of the snowball, but inversely proportional to the speed.[1] What is the acceleration after the snowball has been rolling for a very long time?

Solution

1. After the snowball moves a vertical distance h, it acquires kinetic energy $E = m_0 gh$. This energy will be partly associated with its linear velocity v down the slope, and partly associated with its angular velocity $\omega = v/r_0$. The linear and rotational kinetic energy are

$$E_{\text{lin}} + E_{\text{rot}} = \frac{1}{2}m_0v^2 + \frac{2}{5}m_0r_0^2 \cdot \omega^2 = \left(\frac{1}{2} + \frac{2}{5}\right)m_0v^2 = \frac{9}{10}m_0v^2.$$

From $E = E_{\text{lin}} + E_{\text{rot}}$, we deduce that

$$v^2 = \frac{10}{9}gh.$$

Using the trigonometric relation $\sin\theta = h/s$, and the kinematic formula $v^2 = 2as$, we obtain the desired relation

$$v^2 = \frac{10}{9}gs\sin\theta = 2as \quad\Longrightarrow\quad a_0 = \frac{5}{9}g\sin\theta.$$

We can view $F = m_0a_0$ as the effective gravitational force, taking rolling into account. More generally, if the snowball has mass m, the effective gravitational force is $F = ma_0$.

2. Now the snowball is increasing in mass, and we should apply the impulse formula. The change in momentum is

$$\Delta(mv) = (m + \Delta m)(v + \Delta v) - mv = (m + \Delta m)\Delta v + v\Delta m.$$

This equals the *average force* applied over the time Δt, or

$$(m + \Delta m)\Delta v + v\Delta m = F_{\text{avg}}\Delta t \quad\Longrightarrow\quad F_{\text{avg}} = (m + \Delta m)a + v\frac{\Delta m}{\Delta t}.$$

Finally, the average force is just the average of the force applied at the start of the time period (to the snowball of mass m) and the end (mass $m + \Delta m$). We can use the results from the last question, since these were independent of the mass of the snowball:

$$F_{\text{avg}} = \frac{m + (m + \Delta)}{2}a_0 = (m + \Delta m)a + v\frac{\Delta m}{\Delta t}.$$

[1] A rolling stone gathers no moss.

Rearranging gives

$$(m + \Delta m)a = \left(m + \frac{1}{2}\Delta m\right) a_0 - v\frac{\Delta m}{\Delta t}.$$

3. As Δt goes to zero, Δm gets very small. It follows that

$$(m + \Delta m)a \to ma, \quad \left(m + \frac{1}{2}\Delta m\right) a_0 \to ma_0.$$

On the other hand, we can't neglect $\Delta m/\Delta t$ since both the numerator and denominator are small. Dividing both sides by m, we find

$$a = a_0 - \frac{v}{m}\frac{\Delta m}{\Delta t}.$$

4. The bowling ball undergoes a linear acceleration a_0 down the slope. Taking both $v, m > 0$, a is smaller than a_0 when $\Delta m/\Delta t > 0$, i.e. the snowball is getting heavier. The bowling ball wins if the snowball is gathering snow! On the other hand, if the snowball melts in the sun and loses mass, with $\Delta m/\Delta t < 0$, then $a > a_0$ and the snowball will win. It acts just like a rocket, discarding mass to boost its velocity.

5. Since $m = \rho V = \rho(4/3)\pi r^3$, and the surface area $S = 4\pi r^2$, we can write

$$4\pi\alpha r^2 = 4\pi\alpha \left[\frac{3m}{4\pi}\right]^{2/3} = \beta m^{2/3},$$

where β is a constant defined by the equation. Substituting this into the rate of change of the snowball's mass, we get

$$\frac{\Delta m}{\Delta t} = \frac{\alpha S}{v} = \frac{\beta m^{2/3}}{v}.$$

Then the expression from the last problem gives

$$a = a_0 - \frac{v}{m}\frac{\Delta m}{\Delta t} = a_0 - \frac{v}{m} \cdot \frac{\beta m^{2/3}}{v} = a_0 - \frac{\beta}{m^{1/3}}.$$

As m gets large, the second term gets very small, and eventually $a \approx a_0$. The change in mass makes a negligible contribution to the acceleration!

1.3 Evel Knievel and the crocodile pit

Evel Knievel rides his stunt motorcycle over a semicircular ramp of radius R. He is planning to use this ramp to shoot his motorbike over a pit of ravenous Alabama crocodiles, of length L, immediately after the ramp. His motorcycle can achieve a maximum speed of v, and for simplicity, we assume Knievel can accelerate to this speed instantaneously and at will.

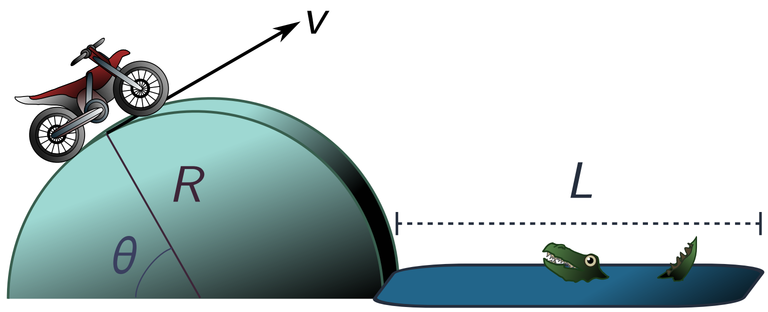


Figure 3: Evil Knievel jumping over a pit of crocodiles.

1. Label the angle from the horizontal by θ. What condition must v satisfy to launch Knievel at an angle θ?
2. Show that if Knievel launches at angle θ, his airtime is

$$t = \frac{1}{g}\left[vc_\theta + \sqrt{v^2c_\theta^2 + 2gRs_\theta}\right],$$

   where $s_\theta = \sin\theta$ and $c_\theta = \cos\theta$.
3. Deduce that after launching at θ, his range over the crocodile pit is

$$r = \frac{v^2 s_\theta}{g}\left[c_\theta + \sqrt{c_\theta^2 + \frac{2gRs_\theta}{v^2}}\right] - R(1 + c_\theta).$$

4. The range is a very unpleasant function to optimise. Instead, let's study a special case. Suppose that Knievel launches horizontally at the top of the ramp with $\theta = \pi/2$. What does v need to be to clear the crocodile pit?
5. For $\theta = \pi/2$, use part (1) to demonstrate that he will *automatically* clear the pit provided

$$(\sqrt{2} - 1)R > L.$$

Solution

1. The centripetal acceleration to keep Knievel on the ramp is just the component of gravity directed towards its centre,

$$a = g\sin\theta.$$

   At speed v, the effective centrifugal acceleration is v^2/R. Thus, the motorbike will launch at angle θ if

$$v^2 > Rg\sin\theta.$$

2. If Knievel launches at angle θ, his velocity has components

$$(v_x, v_y) = v(s_\theta, c_\theta)$$

where $s_\theta = \sin\theta$ and $c_\theta = \cos\theta$. Since his initial height is $h_0 = R\sin\theta$, the height as a function of time is

$$h = h_0 + v_y t - \frac{1}{2}gt^2 = Rs_\theta + vc_\theta t - \frac{1}{2}gt^2.$$

This is a quadratic equation, so we can find the time t to hit the ground by solving $h = 0$:

$$t = \frac{1}{2a}\left[-b \pm \sqrt{b^2 - 4ac}\right] = \frac{1}{g}\left[vc_\theta \mp \sqrt{v^2c_\theta^2 + 2gRs_\theta}\right].$$

To get a positive time, we choose the $+$ symbol:

$$t = \frac{1}{g}\left[vc_\theta + \sqrt{v^2c_\theta^2 + 2gRs_\theta}\right].$$

3. If Knievel launches at angle θ, he still has to cover a horizontal distance $R(1+c_\theta)$ to reach the edge of the ramp. Thus, his range over the crocodile pit is

$$r = vs_\theta t - R(1+c_\theta) = \frac{v^2 s_\theta}{g}\left[c_\theta + \sqrt{c_\theta^2 + \frac{2gRs_\theta}{v^2}}\right] - R(1+c_\theta).$$

4. Consider $\theta = \pi/2$, so that $c_\theta = 0$ and $s_\theta = 1$. Then from the previous question, the range is

$$r = \frac{v^2}{g}\sqrt{\frac{2gR}{v^2}} - R = v\sqrt{\frac{2R}{g}} - R.$$

Thus, Knievel clears the pit provided he launches with speed

$$v\sqrt{\frac{2R}{g}} - R > L \quad \Longrightarrow \quad v > \sqrt{\frac{g}{2R}}(R+L).$$

5. The condition that he launches at all is $v^2 > Rgs_\theta = Rg$. Even if he goes at the minimum speed, he is guaranteed to clear the crocodiles as long as

$$Rg > \frac{g}{2R}(R+L)^2 \quad \Longrightarrow \quad \sqrt{2}R > R+L \quad \Longrightarrow \quad (\sqrt{2}-1)R > L.$$

2 Dimensional analysis and Fermi problems

2.1 Tsunamis and shallow water

Ocean waves behaves rather differently in deep and shallow water. From dimensional analysis, we can learn a little about these differences, and deduce that waves increase in height as they approach the shore. This phenomenon, called *shoaling*, is responsible for tsunamis.

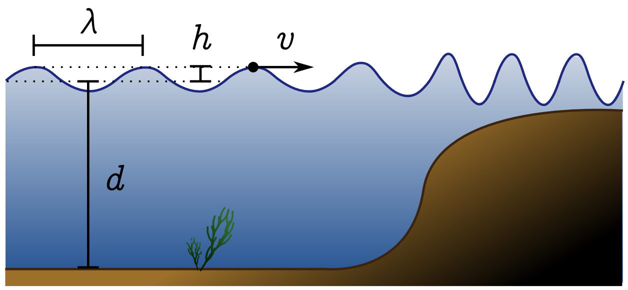


Figure 4: Ocean waves. As the water gets shallower, the waves increase in height.

1. Let λ denote the wavelength of an ocean wave and d the depth of the water. Typically, both are much larger than the height h of the wave, so we can ignore it for the time being. Argue from dimensional analysis that in the *deep water limit* $\lambda \ll d$, the velocity of the wave is proportional to the square root of the wavelength:

$$v \approx \sqrt{g\lambda}.$$

   In the *shallow water limit* $\lambda \gg d$, explain why you expect

$$v \approx \sqrt{gd}.$$

2. Ocean waves can be generated by oscillations beneath the ocean floor. For a source of frequency f, what is the wavelength of the corresponding wave in shallow water? Estimate the wavelength if the source is an earthquake of period $T = 20\,\text{min}$ at depth $d = 4\,\text{km}$, and check your answer is consistent with the shallow water limit.

3. Consider an ocean wave of height h and width w. The energy E carried by a single "cycle" of the wave equals the volume V of water above the mean water level d, multiplied by the gravitational energy density ϵ. By performing a dimensional analysis on each term separately, argue that the total energy in a cycle is approximately

$$E \approx V\epsilon \approx \rho g \lambda w h^2,$$

   where $\rho \approx 10^3\,\text{kg}\,\text{m}^{-3}$ is the density of water and g the gravitational acceleration.

4. Energy in waves is generally *conserved*: the factor E is constant, even as the wavelength λ and height h of the wave change. (We will ignore spreading of the wave.) By applying energy conservation to shallow waves, deduce *Green's law*:

$$h \propto \frac{1}{d^{1/4}}.$$

The increase in height is called *shoaling*. The relation breaks down near shore when the depth d becomes comparable to the height h.

5. Our earthquake from earlier creates a tsunami of height $h_0 = 0.5\,\mathrm{m}$. What is the height, speed, and power per unit width of the tsunami close to the shore? (By "close to the shore", we mean at $h \approx d$ where Green's law breaks down.) You may assume the shallow water equation holds.[2]

Solution

1. Let's write the dimensions of g and ρ in terms M, L, T:

$$[g] = \frac{L}{T^2}, \quad [\rho] = \frac{M}{L^3}.$$

In deep water $d \gg \lambda$, the wave cannot "see" the bottom of the ocean; it is too far away. We only expect the smaller length λ to control the speed. To find the velocity v with dimensions $[v] = L/T$, we can combine g, ρ and λ in precisely one way:

$$v = \sqrt{g\lambda}.$$

It turns out that ρ is not involved! There is no other term to cancel the dimension of mass. Similarly, in shallow water $d \ll \lambda$, the depth is more important than the wavelength, so that we instead get

$$v = \sqrt{gd}.$$

2. The velocity is related to the wavelength and frequency by $v = f\lambda$. Hence, the wavelength of a wave in shallow water of depth d is fixed by question (2):

$$\lambda = \frac{v}{f} = \frac{\sqrt{gd}}{f}.$$

Let's plug in the numbers for the earthquake, noting that $f = 1/T$:

$$\lambda = \sqrt{9.8 \cdot 4000} \cdot (20 \cdot 60)\,\mathrm{m} \approx 237\,\mathrm{km}.$$

This is much larger than than the depth of the ocean, so we can consistently use the shallow water limit.

3. For simplicity, we treat one cycle of the wave as a box, whose volume is the product of length, width and height:

$$V \approx hw\lambda.$$

If E is the dimension of energy, then ϵ has dimensions $[\epsilon] = E/L^3$. Since the energy is due to the gravitational potential of the portion of water above the mean water level, it will

[2] It doesn't quite. We actually need to use the full formula for speed, $v = \sqrt{(g\lambda/2\pi)\tanh(2\pi d/\lambda)^{-1}}$, if we want to make an accurate estimate. But here, the shallow water equation will suffice to get the correct order of magnitude.

involve the height h, the density ρ, and the gravitational acceleration g. The gravitational potential energy is mgh, so the energy *density* should be

$$\epsilon \sim \rho g h.$$

We can get the same answer from dimensional analysis, since

$$[\epsilon] = \frac{E}{L^3} = \frac{ML^2}{L^3T^2} = \frac{M}{L^3} \cdot \frac{L}{T^2} \cdot L = [\rho g h],$$

where we used the fact that $E = ML^2T^{-2}$ (using the formula for kinetic energy, for example). Thus, the energy carried by one cycle of the wave is

$$E \approx V\epsilon \approx \rho g \lambda w h^2.$$

4. In shallow waves, question (3) shows that $\lambda \propto \sqrt{d}$. Since ρ, g, w are constant, we have

$$E \propto \sqrt{d} h^2.$$

Taking the square root, and using the fact that E is constant, we obtain Green's law:

$$hd^{1/4} \propto \sqrt{E} \implies h \propto \frac{1}{d^{1/4}}.$$

5. The wave is "close to shore" when the height is comparable to the depth of the water, $h \approx d$. We can use this, along with Green's law and the initial height and depth, to determine h:

$$hd^{1/4} = h^{5/4} = h_0 d_0^{1/4} \implies h = h_0^{4/5} d_0^{1/5} = 4000^{1/5} \approx 5.25\,\mathrm{m}.$$

Assuming the shallow water equation holds,

$$v \approx \sqrt{gd} \approx \sqrt{9.8 \cdot 5.25}\,\mathrm{m} \cdot \mathrm{s}^{-1} = 7.2\,\mathrm{m} \cdot \mathrm{s}^{-1}.$$

The tsunami is around 5 meters high and travelling at a velocity of $7\,\mathrm{m} \cdot \mathrm{s}^{-1}$. This doesn't sound that high or fast, but is more than enough to cause catastrophic damage.

To see how much energy such a tsunami delivers, we use our expression from part (3). To find the total power, we divide the energy delivered per wave E by the period of the wave, $T = 20\,\mathrm{min}$. To find the power P per unit width, we divide by w. The result is

$$P = \frac{E}{Tw} \approx \frac{\rho g \lambda h^2}{T} = \rho g v h^2,$$

using $v = \lambda/T$. To evaluate this, we plug in the value for v we calculate, and the density of water $\rho \approx 10^3\,\mathrm{kg} \cdot \mathrm{m}^{-3}$. This gives

$$P \approx 10^3 \cdot 9.8 \cdot 7.2 \cdot 5.25^2\,\mathrm{W} \cdot \mathrm{m}^{-1} \sim 1\,\mathrm{MW} \cdot \mathrm{m}^{-1}.$$

The tsunami delivers around 1 megawatt per meter of shoreline. This is enough to power several hundred households! Since the tsunami is supplying this amount *for each metre of shoreline*, it's not too hard to see why a tsunami of modest height can still wreak terrible havoc.

References

- "The shallow water wave equation and tsunami propagation" (2011). Terry Tao.

2.2 Turbulence in a tea cup

Stir a cup of coffee vigorously enough, and the fluid will begin to mix in a chaotic or *turbulent* way. Unlike the steady flow of water through a pipe, the behaviour of turbulent fluids is unpredictable and poorly understood. However, for many purposes, we can do suprisingly well by modelling a turbulent fluid as a collection of (three-dimensional) eddies of different sizes, with larger eddies feeding into smaller ones and losing energy in the process.

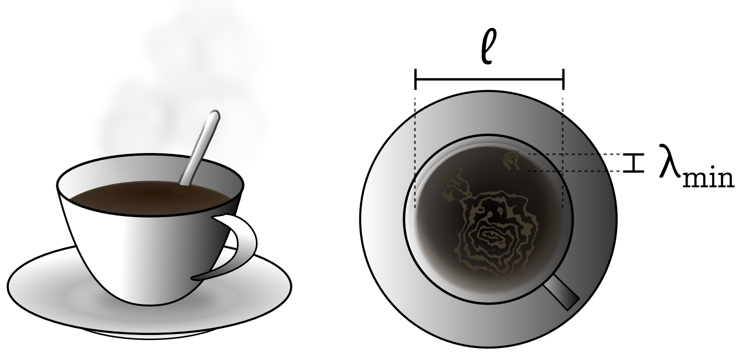


Figure 5: A well-stirred cup of coffee. On the right, a large eddy (size $\sim \ell$) and the smallest eddy (size λ_{min}) are depicted.

Suppose our cup of coffee has characteristic length ℓ, and the coffee has density ρ. When it is turbulently mixed, the largest eddies will be a similar size to the cup, order ℓ, and experience fluctuations in velocity of size Δv due to interaction with other eddies. The fluid also has internal drag[3] or *viscosity* η, with units $\mathrm{N} \cdot \mathrm{s}/\mathrm{m}^2$.

1. Let ϵ be the rate at which kinetic energy dissipates per unit mass due to eddies. Observation shows that this energy loss is independent of the fluid's viscosity. Argue on dimensional grounds that
$$\epsilon \approx \frac{(\Delta v)^3}{\ell}.$$
Why doesn't the density ρ appear?

2. Kinetic energy can also be lost due to internal friction. Argue that the time scale for this dissipation due to viscosity is
$$\tau_{\text{drag}} \approx \frac{\ell^2 \rho}{\mu}.$$

3. Using the previous two questions, show that eddy losses[4] dominate viscosity losses provided
$$\frac{\ell \rho \Delta v}{\mu} \gg 1.$$

[3]More precisely, viscosity is the resistance to *shear flows*. A simple way to create shear flow is by moving a large plate along the surface of a stationary fluid. Experiments show that the friction per unit area of plate is proportional to the speed we move it, and inversely proportional to the height; the proportionality constant at unit height is the viscosity. Since layers of fluid also generate shear flows, viscosity creates internal friction.

[4]Since ϵ depends on $\ell, \Delta v$, you need not consider it when finding the time scale for eddy losses.

The quantity on the left is called the *Reynolds number*, $\mathrm{Re} = \ell\rho\Delta v/mu$. In fact, one *definition* of turbulence is fluid flow where the Reynolds number is high.

4. So far, we have focused on the largest eddies. These feed energy into smaller eddies of size λ and velocity uncertainty Δv_λ, which have an associated *eddy Reynolds number*,

$$\mathrm{Re}_\lambda = \frac{\lambda\rho\Delta v_\lambda}{\mu}.$$

When the eddy Reynolds number is less than 1, eddies of the corresponding size are prevented from forming by viscosity.[5] Surprisingly, the rate of energy dissipation per unit mass in these smaller eddies is ϵ, the same as the larger eddies.[6] Show from dimensional analysis that the minimum eddy size is roughly

$$\lambda_{\min} \approx \left(\frac{\mu^3}{\epsilon\rho^3}\right)^{1/4}.$$

5. If a cup of coffee is stirred violently to Reynolds number $\mathrm{Re} \approx 10^4$, estimate the size of the smallest eddies in the cup.

Solution

1. Let $[\cdot]$ denote the dimensions of a physical quantity, and M, L, T mass, length and time respectively. Then energy per unit mass per unit time has dimension

$$[\epsilon] = \frac{\text{energy}}{MT} = \frac{M(L/T)^2}{MT} = \frac{L^2}{T^3},$$

where we can remember the dimension for energy using kinetic energy, $K = mv^2/2$. (The dimension does not depend on what form of energy we look at.) The dimensions for the remaining physical quantities are easier:

$$[\ell] = L, \quad [\rho] = \frac{M}{L^3}, \quad [\Delta v] = \frac{L}{T}.$$

Since mass does not appear in $[\epsilon]$, and the viscosity is not involved in this type of dissipation, the density ρ cannot appear since there is nothing besides μ to cancel the mass units. We can easily combine ℓ and Δv to get something with the correct dimension, and deduce an approximate relationship between $\epsilon, \Delta v$ and ℓ:

$$\frac{[(\Delta v)^3]}{[\ell]} = \frac{L^3}{LT^3} = [\epsilon] \quad \Longrightarrow \quad \epsilon \approx \frac{(\Delta v)^3}{\ell}.$$

[5] Lewis Fry Richardson not only invented the eddy model, but this brilliant mnemonic couplet: "Big whirls have little whirls that feed on their velocity, and little whirls have lesser whirls and so on to viscosity."

[6] This is not at all obvious, but roughly, follows because we can fit more small eddies in the container. Intriguingly, this makes the turbulent fluid like a *fractal*: the structure of eddies repeats itself as we zoom in, until viscosity begins to play a role. At infinite Reynolds number, it really is a fractal!

2. Viscosity has dimensions

$$[\mu] = \frac{[\mathrm{N}][\mathrm{s}]}{[\mathrm{m}^2]} = \frac{MLT}{T^2L^2} = \frac{M}{LT}.$$

We can combine with ρ and ℓ to get something with the dimensions of time; Δv is not involved since friction is independent of the eddies. The unique combination with the right units is

$$\frac{[\ell^2\rho]}{[\mu]} = \frac{L^2 \cdot M \cdot LT}{L^3 \cdot M} = T \quad \Longrightarrow \quad \tau_{\mathrm{drag}} = \frac{\ell^2\rho}{\mu}.$$

3. Returning to eddy losses, its easy to cook up a time scale from the basic physical quantities ℓ and Δv:

$$\tau_{\mathrm{eddy}} \approx \frac{\ell}{\Delta v}.$$

In order for dissipation of energy by the eddies to dominate, we require $\tau_{\mathrm{eddy}} \ll \tau_{\mathrm{drag}}$, that is, energy is much more quickly dissipated by the eddies than by friction. Comparing the two expressions, we find

$$\frac{\ell}{\Delta v} \ll \frac{\ell^2\rho}{\mu} \quad \Longrightarrow \quad \frac{\ell\rho\Delta v}{\mu} = \mathrm{Re} \gg 1.$$

4. By assumption, the rate of energy dissipation ϵ is the same for all eddies, so the reasoning in part (1) gives $\epsilon \approx (\Delta v_\lambda)^3/\lambda$. Rearranging, we have $\Delta v_\lambda \approx (\epsilon\lambda)^{1/3}$. We now set $\mathrm{Re}_\lambda = 1$ and solve for the minimum eddy size λ_{min}:

$$1 = \mathrm{Re}_\lambda = \frac{\lambda\rho\Delta v_\lambda}{\mu} \approx \frac{\lambda^{4/3}\epsilon^{1/3}\rho}{\mu} \quad \Longrightarrow \quad \lambda_{\mathrm{min}} \approx \left(\frac{\mu}{\epsilon^{1/3}\rho}\right)^{3/4} = \left(\frac{\mu^3}{\epsilon\rho^3}\right)^{1/4}.$$

5. There is a cute shortcut here. First, the previous question tells us how Re_λ scales with λ:

$$\mathrm{Re}_\lambda \approx \frac{\epsilon^{1/3}\rho\lambda^{4/3}}{\mu} = \alpha\lambda^{4/3},$$

where α is a constant independent of λ. But the Reynolds number is simply the eddy Reynolds number for $\lambda = \ell$, $\mathrm{Re} = \mathrm{Re}_\ell$, and the eddy Reynolds number is unity for the smallest eddies. Hence,

$$\mathrm{Re}_{\lambda_{\mathrm{min}}} \approx \alpha\lambda_{\mathrm{min}}^{4/3} = 1, \quad \mathrm{Re} \approx \alpha\ell^{4/3} \quad \Longrightarrow \quad \lambda_{\mathrm{min}}^{4/3} \approx \frac{\ell^{4/3}}{\mathrm{Re}}.$$

For our turbulent coffee, $\ell \approx 10\,\mathrm{cm}$ and $\mathrm{Re} \approx 10^4$, so we estimate a minimum eddy size

$$\lambda_{\mathrm{min}} \approx \frac{\ell}{\mathrm{Re}^{3/4}} \approx \frac{10\,\mathrm{cm}}{10^3} = 0.1\,\mathrm{mm}.$$

References

- *Microphysics of clouds and precipitation* (2010). H. R. Pruppacher, J. D. Klett.

Figure 6: Our cast of characters.

2.3 A Fermi free-for-all

Order of magnitude approximations, or *Fermi estimates*, are a fun and surprisingly powerful approach to solving problems. Here, we offer a medley of Fermi problems, ranging from Starbucks to stars to sneezes. There are a few techniques you may find useful:

- taking the *geometric mean* $\sqrt{UL}$ of upper and lower guesses U and L for a quantity;
- *factorising* your answer into a string of subestimates with intermediate units; and
- using *dimensional analysis* and *simple physics*.[7]

You may also need data about the world (supplied) and common sense (not included).

1. We start with big numbers, and answer an age old question: are there more stars in the sky, or grains of sand? And what about atoms in a single grain?

   (a) **Stars.** How many stars are there in the observable universe?
   Data. Astronomers count roughly 100 billion galaxies. Small dwarf galaxies have on the order of 100 million stars, while massive elliptical galaxies can have in excess of 10 trillion stars.

   (b) **Sand.** Estimate the number of grains of sand on all the beaches of the world.
   Data. Sand particles range in size from 0.0625 mm to 2 mm. The earth has 620,000 km of coastline.

   (c) **Atoms.** How many atoms are in an average grain of sand? Compare to the two previous numbers, and comment on your result.
   Data. Sand is made out of silicon dioxide SiO_2, with molar mass 60 g. Avogadro's constant is $N_A = 6 \times 10^{23}$.

[7] "Simple physics" means to solve a caricature of the problem, where you ignore everything but the most important mechanism.

2. Next, we add some physics into the mix.

   (a) **Raindrops.**

      i. Using dimensional analysis, estimate the size of a raindrop.

      ii. As they bang into each other, raindrops resonate like the head of a drum, since they are under tension. What is the approximate frequency of this resonance?

   Data. The surface tension of water is $\sigma = 0.07$ N/m, with dimensions $[\sigma] = M/T^2$.[8] Surface tension wants to make the raindrop small; gravity wants to spread it out.

   (b) **Sneezes.** Here's a sillier one.

      i. How much force is released in the average sneeze? No dimensional analysis required, just regular Newtonian mechanics.

      ii. How many people are required to sneeze a 1 kg cubesat[9] into space?

   Data. The lung capacity of an adult is around 5 L, and sneezes are emitted with a final velocity of roughly 50 m/s. Launch velocity at the earth's surface is 11 km/s.

3. We end with some harder "real life" Fermi estimates.

   (a) **Hungarian GDP.** Guess the size of Hungary's economy, measured by GDP.[10]

   Data. Canada's GDP is 1.6 trillion USD. India's GDP is 2.6 trillion USD.

   (b) **Starbucks.** Estimate the number of Starbucks stores in Seattle.

   Data. Seattle city has a population of around 700,000.

Solution.

1. (a) A simple approach is to multiply the total number of galaxies by the "average" number of stars per galaxy:

$$\frac{\text{stars}}{\text{universe}} = \frac{\text{stars}}{\text{galaxy}} \times \frac{\text{galaxies}}{\text{universe}}.$$

Of course, we don't know the average number of stars exactly, but a useful trick we will use again and again is to take the geometric mean of upper and lower guesses. In this case, the lower guess is a dwarf galaxy, and the upper guess an elliptical galaxy:

$$\frac{\text{stars}}{\text{galaxy}} \sim \sqrt{100 \times 10^6 \times 10^{13}} \approx 3 \times 10^{10}.$$

This gives

$$\frac{\text{stars}}{\text{universe}} = \frac{\text{stars}}{\text{galaxy}} \times \frac{\text{galaxies}}{\text{universe}} \sim 3 \times 10^{10} \times 10^{11} \sim 3 \times 10^{21} \text{ stars}.$$

We estimate there are around 3×10^{21} stars in the universe!

[8]Concretely, if I try and cut water with a knife, there is a resistance of 70 mN per metre of knife.

[9]A cubesat is a small, cubical satellite.

[10]*Gross domestic product.* This is the total monetary worth of all goods and services produced in a year, conventionally reported in US dollars.

(b) In this case, we're going to estimate the volume of beach in the world and divide by the volume of the average grain of sand. Since this problem can be treat in a more conventional way, we use more conventional notation. We guess that the volume of beach is the total length of the world's coastline, multiplied by the percentage which has beach, multiplied by the average width and depth of a beach:

$$V_{\text{beach}} = L_{\text{coastline}} \times p_{\text{beach}} \times w_{\text{beach}} \times d_{\text{beach}}.$$

We have the coastline, but everything else we have to estimate. A lot of the world's coastline is beach, but some of it has cliffs, rocks, ice, etc. Let's set $p_{\text{beach}} = 70\%$. What about the width and depth of an average beach? Of course the profile will vary, but a small beach might be a few meters, and a large beach 50 meters, so we will take the mean:

$$w_{\text{beach}} \sim \sqrt{1 \times 50} \text{ m} \approx 7 \text{ m}.$$

Finally, from the way beaches grade into the water, I would guess the depth of sand is usually a couple of meters, say $d_{\text{beach}} \sim 2$ m. This gives

$$V_{\text{beach}} \sim 620,000 \text{ km} \times 0.7 \times 7 \text{ m} \times 2 \text{ m} \approx 6.2 \times 10^9 \text{ m}^3.$$

Now for the grain of sand. We've been given the official geological lower and upper bound on grain size, so we just take the geometric mean and hope this is about the average grain size:

$$d_{\text{grain}} \sim \sqrt{0.0625 \times 2} \text{ mm} \approx 0.35 \text{ mm}.$$

Assuming (for simplicity) that grains are tiny cubes, we guess the average volume us

$$V_{\text{grain}} \sim 0.35^3 \text{ mm}^3 \approx 4.3 \times 10^{-10} \text{ m}^3.$$

The number of grains of sand is then estimated to be

$$\frac{V_{\text{beach}}}{V_{\text{grain}}} \sim \frac{6.2 \times 10^9 \text{ m}^3}{4.3 \times 10^{-10} \text{ m}^3} \approx 1.4 \times 10^{19}.$$

That's a lot, but falls a few orders of magnitude short of stars in the universe!

(c) We already calculated V_{grain} in the last problem. To find its mass, we can estimate the density. Sand sinks in water, so we know it has to be heavier than water, let's say twice as heavy. Then the mass of an average grain is

$$m_{\text{grain}} \sim 2\rho_{\text{water}} \times V_{\text{grain}} \sim (2 \times 10^{-3} \text{ g/mm}^3) \times 0.43 \text{ mm}^3 \approx 8 \times 10^{-3} \text{ g}.$$

Now we divide by the molar mass to learn how many moles of SiO_2 are in there:

$$N_{\text{SiO}_2} = \frac{m_{\text{grain}}}{M_{\text{SiO}_2}} \sim \frac{8 \times 10^{-3} \text{ g}}{60 \text{ g}} \approx 1.3 \times 10^{-4}.$$

Since there are three atoms in each silicon dioxide molecule, the total number of atoms is

$$3 \times N_{\text{SiO}_2} \times N_A \sim 3 \times (1.3 \times 10^{-4}) \times (6 \times 10^{23}) \approx 3 \times 10^{20}.$$

There are more atoms in a grain of sand than grains of sand on the world's beaches. And a handful of sand has more atoms than stars in the universe!

2. (a) i. Here are a few facts we will need:
   - the *surface gravity* of earth is $g \approx 10$ m/s^2, with dimensions $[g] = L/T^2$;
   - water has a density $\rho \approx 10^3$ kg/m^3 with dimension $[\rho] = M/L^3$;
   - the surface tension has dimensions $[\sigma] = M/T^2$.

   There is only one way of combining these parameters to get something with the dimensions of length. We divide the surface tension by the water density to cancel out the mass dimension M, then divide by surface gravity to cancel the dimension of time:

$$\left[\frac{\sigma}{\rho}\right] = \frac{M/T^2}{M/L^3} = \frac{L^3}{T^2}, \quad \left[\frac{\sigma}{\rho} \cdot \frac{1}{g}\right] = \frac{L^3/T^2}{L/T^2} = L^2.$$

   This has the dimensions of length squared, so we take the square root and get a guess for the size of a rain droplet:

$$\ell \sim \frac{\sigma}{\rho g} \approx \sqrt{\frac{0.07}{10^3 \times 10}} \approx 2.6 \text{ mm}.$$

   Raindrops have different sizes, but apparently "medium size" raindrops are in the range 1.7–3.2 mm. Our guess is correct!

   ii. Now we estimate the frequency of vibrating raindrops. Gravity has nothing to do with the vibration of drumhead, and similarly, for the vibration of a raindrop it plays no role. Our goal is to obtain something with dimensions of frequency, $1/T$, using properties of the raindrop itself. Since the units of surface tension are M/T^2, if we divide by the mass of the raindrop, $m \sim (4\pi/3)\rho r^3$, we will get something with units $1/T^2$, where r is the radius we just determined. The *square root* will have the correct dimensions! We therefore guess that

$$f \sim \sqrt{\frac{\sigma}{m}} \approx \sqrt{\frac{0.07}{1000 \times (4\pi/3)(0.0026)^3}} \approx 30 \text{ Hz}.$$

   So clashing raindrops should vibrate once every couple of seconds.

(b) i. The volume of air released in a sneeze varies, but for a lung capacity of 5 L, the average sneeze probably releases a parcel of air with volume ~ 2 L. The density of air is $\rho = 1$ kg/m^3, so the mass of the air released is

$$m = \rho V \sim (1 \text{ kg/m}^3) \times 2 \text{ L} = (1 \text{ kg/m}^3) \times 2 \times 10^{-3} \text{ m}^3 = 2 \text{ g}.$$

   A sneeze lasts maybe half a second, $\Delta t \approx 0.5$ s. If the final velocity is around $v \approx 50$ m/s, then the formula for impulse gives us the average force:

$$F = \frac{p_{\text{sneeze}}}{} = \frac{mv}{\Delta t} = \frac{2 \text{ g} \times 50 \text{ m/s}}{0.5 \text{ s}} \approx 0.2 \text{ N}.$$

   ii. To release a 1 kg cubesat into space, it needs momentum

$$p_{\text{cubesat}} = m_{\text{cubesat}} v_{\text{escape}} = 11 \times 10^3 \text{ kg} \cdot \text{m/s}.$$

Assuming all the linear momentum of the sneeze can somehow be imparted to the cubesat (unlikely, but we don't actually care about the logistics of our sneeze launcher), the number of sneezers required is just

$$N \sim \frac{p_{\text{cubesat}}}{p_{\text{sneeze}}} = \frac{11 \times 10^3}{0.1} \approx 10^5 \text{ people.}$$

It looks like we would need around 100,000 people to sneeze the cubesat into space.

3. (a) One approach is to factorise using people as an intermediate unit:

$$\frac{\text{GDP}}{\text{Hungary}} = \frac{\text{GDP}}{\text{person}} \times \frac{\text{people}}{\text{Hungary}}.$$

The second factor is just the population of Hungary. You probably know that Hungary is a small country in Eastern Europe, so it seems reasonable to guess it's population is half of Canada's (40 million):

$$\frac{\text{people}}{\text{Hungary}} \sim 20 \text{ million.}$$

The first factor is the GDP per capita, which is correlated roughly with the standard of living in a country. This is where the data given become useful! Canada has a very high standard of living; India has a much lower standard of living. Hungary is probably somewhere in between. We will therefore try to take a geometric average of the GDP per capita for Canada and India:

$$\frac{\text{Hungary GDP}}{\text{Hungary population}} \sim \sqrt{\frac{\text{Canada GDP}}{\text{Canada population}} \times \frac{\text{India GDP}}{\text{India population}}}.$$

You have been given the GDP of Canada and India, but we have to input the populations. Canada, as we've said, it roughly 40 million, while India, like China, has a huge population of around 1 billion. So we plug in the numbers to find

$$\frac{\text{Hungary GDP}}{\text{Hungary population}} \sim \text{USD}\$10,000.$$

Our final estimate is

$$\begin{aligned}\frac{\text{GDP}}{\text{Hungary}} &= \frac{\text{GDP}}{\text{person}} \times \frac{\text{people}}{\text{Hungary}} \\ &\sim 20 \times 10^6 \times \text{USD}\$10,000 \\ &\sim \text{USD}\$200 \text{ billion.}\end{aligned}$$

The actual GDP of Hungary: USD\$140 billion! Our guess is pretty close.

(b) We'll use a similar approach to the famous piano tuner problem (see forthcoming notes on Fermi estimates). We'll start by introducing the natural intermediate unit of people,

$$\frac{\text{Starbucks}}{\text{Seattle}} = \frac{\text{Starbucks}}{\text{person}} \times \frac{\text{people}}{\text{Seattle}}.$$

The second factor is just the population of Seattle, which we know is 700,000. To estimate the first factor easier, let's introduce another intermediate unit, namely *orders per day*:

$$\frac{\text{Starbucks}}{\text{person}} = \frac{\text{Starbucks}}{\text{orders}} \times \frac{\text{orders}}{\text{person}}.$$

We'll leave the time frame implicit to cut down on notational clutter. Now, the first factor here is the reciprocal of the number of orders that a Starbucks store processes every day. I'm guessing that at busy times a store could do something like 10 orders a minute, and at slow times around 1 order a minute. The geometric mean is $\sqrt{10 \times 1} \approx 3$ orders a minute, so over an 8 hour day, this leads to a total number of orders

$$\frac{\text{orders}}{\text{Starbucks}} \sim 8 \times 60 \times 3 = 1440.$$

What about orders per person? In Seattle, I guess maybe 1 in 3 people order a coffee each day (again, I can get this averaging 1 and 10). We are led to the final guess

$$\begin{aligned} \frac{\text{Starbucks}}{\text{Seattle}} &\sim \frac{\text{Starbucks}}{\text{order}} \times \frac{\text{order}}{\text{person}} \times \frac{\text{people}}{\text{Seattle}} \\ &\approx \frac{1}{1440} \times \frac{1}{3} \times 700,000 \approx 160 \text{ stores.} \end{aligned}$$

According to Statista, Seattle has 142 Starbucks stores. We're pretty close!

3 Gravity

3.1 Getting a lift into space

A *space elevator* is a giant cable suspended between the earth and an orbiting counterweight. Both the cable and counterweight are fixed in the rotating reference frame of the earth. The elevator can be used to efficiently transport objects from the surface into orbit, but also as a cheap launchpad.

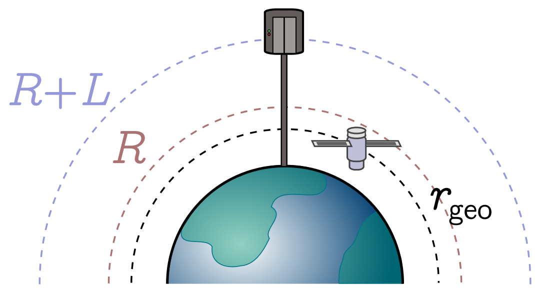


Figure 7: A satellite in geostationary orbit at radius r_{geo}. A space elevator connects a counterweight in low orbit to the surface via a cable of length $2L$. The cable's centre of mass lies at radius R, above r_{geo}.

1. To begin with, forget the cable, and consider a *geostationary* satellite orbiting at a fixed location over the equator.

   - Determine the radius r_{geo} of a geostationary orbit in terms of the mass of the earth M and angular frequency ω about its axis.
   - Confirm that r_{geo} obeys Kepler's third law, i.e. the square of the orbital period is proportional to the cube of the radius.

2. To make the space elevator, we now attach a cable to the satellite. The satellite acts as a counterweight, pulling the cable taut, but needs to move into a higher orbit in order to balance the cable tension. Provided this orbit is high enough, the space elevator will double as a rocket launchpad. Show that objects released from the elevator at $r_{\text{esc}} = 2^{1/3} r_{\text{geo}}$ will be launched into deep space.

3. The dynamics of the elevator itself are complicated, so we will consider a simplified model where the cable is treated as a rigid rod of length $2L$, with all of its mass concentrated at the centre, radius R. The counterweight is therefore at radius $R + L$.

   - Find the exact relationship between L, R, and the earth's mass M and rotational period ω.
   - Assuming $L \ll R$, show that the rod's centre of mass is further out than the geostationary radius r_{geo}. This somewhat counterintuitive result also holds for real space elevator designs! You may use the fact that, for $x \ll 1$,

$$\frac{1}{1+x} \approx 1 - x.$$

Solution

1. The gravitational acceleration is given by Newton's law of gravitation:

$$a = \frac{GM}{r_{\text{geo}}^2}.$$

   The centrifugal acceleration is

$$a = \frac{v^2}{r_{\text{geo}}} = \omega^2 r.$$

   Equating the two, we find

$$r_{\text{geo}}^3 = \frac{GM}{\omega^2}.$$

   Since $\omega \propto 1/T$, where T is the period of the orbit, Kepler's third law is obeyed.

2. Since the whole elevator is geostationary, it rotates with angular frequency ω. At radius r_{esc}, the speed is $v = \omega r_{\text{esc}}$. We recall that the gravitational potential is $U = -GMm/r$. Finally, we can determine r_{esc} by demanding that the total energy vanish:

$$E = U + K = m\left(\frac{1}{2}\omega^2 r_{\text{esc}}^2 - \frac{GM}{r_{\text{esc}}}\right) = 0 \quad \Longrightarrow \quad r_{\text{esc}}^3 = \frac{2GM}{\omega^2} = 2r_{\text{geo}}^3.$$

3. Treat the rod as concentrated at its centre of mass at radius R. In order for the rod and the satellite to have the same angular velocity, we require the forces in the rotating reference frame to balance:

$$\omega^2[(R-L) + (R+L)] = 2R\omega^2 = GM\left[\frac{1}{(R-L)^2} + \frac{1}{(R+L)^2}\right] = \frac{2GM(R^2+L^2)}{(R^2-L^2)^2}.$$

   Rearranging, we find that

$$\frac{GM}{\omega^2} = \frac{R(R^2-L^2)^2}{(R^2+L^2)}.$$

   If $L \ll R$, then $(L/R)^2 \ll 1$ and hence

$$\frac{1}{R^2+L^2} = \frac{1}{R^2(1+L^2/R^2)} \approx \frac{1}{R^2}\left(1 - \frac{L^2}{R^2}\right),$$

   using our approximation $1/(1+x) \approx 1 - x$. It follows that

$$\frac{GM}{\omega^2} \approx \frac{1}{R}(R^2 - 2L^2)(R^2 - L^2) \approx R^3 - 3RL^2.$$

   Comparing to the radius of the geostationary orbit, we find

$$r_{\text{geo}}^3 \approx R^3 - 3RL^2,$$

   which implies that $r_{\text{geo}} < R$.

3.2 Hubble's law and dark energy

If we point a telescope at random in the night sky, we discover something surprising: faraway galaxies and stars are all moving away from us.[11] Even more surprising, the speed v of any object is proportional to its distance d from the earth, with

$$v = Hd.$$

The parameter H is called the *Hubble constant* (though it can in fact change), and the relation between velocity and distance is called *Hubble's law*.

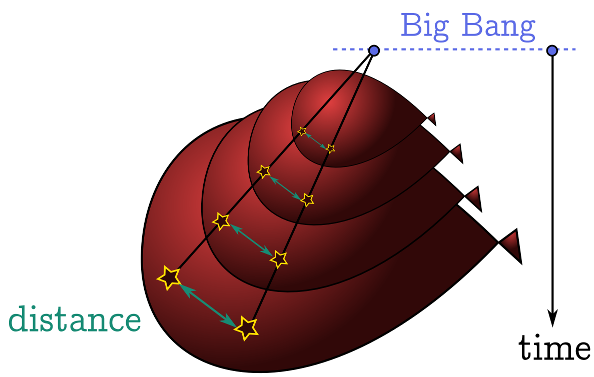


Figure 8: The cosmic balloon, inflated by dark energy.

A simple analogy helps illustrate. Imagine the universe as a balloon, with objects (like the stars in the image above) in a fixed position on the balloon "skin". Both the distance and relative velocity of any two objects will be proportional to the size of the balloon, and hence each other. The constant of proportionality is H.

1. The universe is expanding. Explain why Hubble's law implies that it does so at an *accelerating* rate.

2. The Virgo cluster is around 55 million light years away and receding at a speed of $1200\,\mathrm{km\,s^{-1}}$. By running time backwards, explain why you expect a Big Bang where everything is located at the same point. From the Virgo cluster and Hubble's law, estimate the age of the universe.

3. Since gravity is an attractive force, the continual expansion of the universe is somewhat mysterious. Why doesn't all the mass collapse back in on itself? The answer to this question is *dark energy*. Although we're not entirely sure what dark energy is, we can model it as an energy density ρ due to empty space itself. This energy does not change with time, since the vacuum always looks the same.

[11] How? Well, we know what frequencies of light stars like to emit since they are made of chemicals we find on earth. These frequencies are *Doppler-shifted*, or stretched, if the stars in a galaxy are moving away from us, allowing us to determine the speed of recession. Distance is a bit harder to work out, with different methods needed for different distance scales.

The state-of-the-art description of gravity is Einstein's theory of *general relativity*. For our purposes, all we need to know is that gravitational effects are governed by Newton's constant G and the speed of light c, where

$$G = 6.67 \times 10^{-11}\,\mathrm{N} \cdot \mathrm{m}^2 \cdot \mathrm{kg}^{-2}, \quad c = 3 \times 10^8\,\mathrm{m}\,\mathrm{s}^{-1}.$$

Using dimensional analysis, argue that Hubble's constant is related to dark energy by

$$H^2 = \frac{\eta G \rho}{c^2}$$

for some (dimensionless) number η. This is the *Friedmann equation*.

4. Assuming that $\eta \sim 1$, estimate the dark energy density of the universe.

Solution

1. Hubble's law says that

$$v = Hd.$$

Assuming that H is constant, the rate of change of the left side is just the acceleration a, while the rate of change of the right side is v, multiplied by the constant H. So

$$a = Hv = H^2 d.$$

Since the universe is expanding, d increases with time. Hence, the acceleration also increases with time!

2. Let's run time backwards until a faraway object collides with us. If the distance is d, and the velocity v, then by Hubble's law the time needed to hit us is

$$t_{\text{collision}} = \frac{d}{v} = \frac{1}{H}.$$

Since this is the same for any object, it suggests that a time $t_{\text{collision}}$, every object in the universe was in the same place. This must be the Big Bang! The age of the universe is then $t_{\text{collision}}$, which we can estimate from the Virgo cluster as

$$t_{\text{collision}} = \frac{d}{v} = \frac{53 \times 10^6 \times (3 \times 10^8\,\mathrm{m/s})}{1.2 \times 10^6\,\mathrm{m/s}}\,\text{years} \approx 13.75 \times 10^9\,\text{years}.$$

We guess the universe is about 13.75 billion years old. The current best estimate is 13.80 billion years!

3. We let L, M, T denote the dimensions of length, mass and time respectively. We know from the previous question that H has the units of inverse time, $[H] = T^{-1}$, and the speed of light clearly has dimensions $[c] = L/T$. We can also find the dimensions of G from the dimensions of the Newton:

$$[\mathrm{N}] = \frac{ML}{T^2} \quad \Longrightarrow \quad [G] = \frac{[\mathrm{N}][\mathrm{m}]^2}{[\mathrm{kg}]^2} = \frac{L^3}{T^2 M}.$$

Finally, since the dimensions of energy are $[E] = ML^2/T^2$, the dimensions of energy *density* (energy over volume) are

$$[\rho] = \frac{[E]}{[V]} = \frac{M}{LT^2}.$$

Let's look for an equation of the form

$$H^\alpha = \eta G^\beta c^\gamma \rho^\delta$$

which has dimensions

$$\frac{1}{T^\alpha} = \eta \left(\frac{L^3}{T^2 M}\right)^\beta \left(\frac{L}{T}\right)^\gamma \left(\frac{M}{LT^2}\right)^\delta = \eta \left(\frac{L^{3\beta+\gamma-\delta} M^{\delta-\beta}}{T^{2\beta+\gamma+2\delta}}\right).$$

This looks hard, but there is no mass or length on the LHS so

$$\delta - \beta = 3\beta + \gamma - \delta = 0 \quad \Longrightarrow \quad 2\beta + \gamma = 0.$$

But then, matching powers of time on both sides,

$$\alpha = 2\beta + \gamma + 2\delta = 2\delta.$$

The simplest way to satisfy all of these constraints is $\beta = \delta = 1$ and $\alpha = -\gamma = 2$. This gives us the Friedmann equation

$$H^2 = \frac{\eta G \rho}{c^2}.$$

4. To find the density of dark energy, we can simply invert the Friedmann equation to make ρ the subject, and plug in the age of the universe calculated in part (a):

$$\rho \sim \frac{c^2 H^2}{G} = \frac{(3\times 10^8)^2}{(6.67\times 10^{-11})(13.75\times 10^9 \times 365 \times 24 \times 60^2)^2}\frac{\mathrm{J}}{\mathrm{m}^3} \approx 7\times 10^{-9}\frac{\mathrm{J}}{\mathrm{m}^3}.$$

Doing the full gravity calculation shows that $\eta = 8\pi/3 \sim 10$, so our answer is too large by a factor of around 10. Accounting for this, we guess $\rho \sim 10^{-9}\mathrm{J}/\mathrm{m}^3$, which matches the current best estimate to within an order of magnitude.[12]

[12] In fact, ρ is the *total* energy density of the universe, including things besides dark energy. While dark energy density does not change with time, other forms of energy are diluted as the universe expands; from the Friedmann equation, this means that H changes with time. Indeed, in the past H was very different. However, dark energy constitutes around 70% of the total density, explaining why our estimate here is still reasonably accurate. It also explains why H is approximately constant, at least in the current epoch of expansion.

3.3 Gravitational postal service

Mega-corporation Mammonzon drills a hole through the centre of the earth and sets up an antipodal delivery service, dropping packages directly through to the other side of the world. Your job, as a new Mammonzon employee, is to determine package delivery times! You soon realise that there is a complication: the strength of gravity changes as the package moves through the tunnel. To help out, your boss recommends Newton's *Principa Mathematica*, which provides a marvellous result called the Sphere Theorem:

- an object outside a spherical body (of constant density) is gravitationally attracted to it *as if all the mass were concentrated at the centre*;[13]
- an object inside a spherical shell *feels no gravitational attraction to the shell*.

We can use the Sphere Theorem, and a surprising analogy to springs, to work out the package transit time.

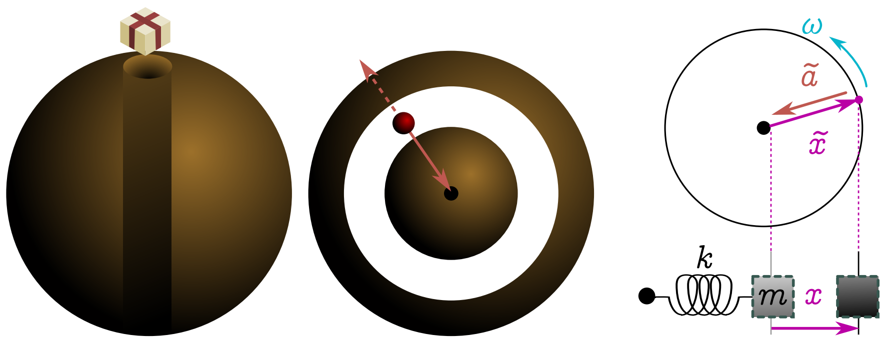


Figure 9: *Left*. A package travelling through the hole in the middle of the earth. *Middle*. The sphere theorem; the red mass feels no attraction to the shell, and attraction to the inner sphere as if all the mass were concentrated at the centre. *Right*. Phasor approach to solving the spring-mass problem.

1. Let r denote the radial distance from the centre of the earth. From the Sphere Theorem, show that a package at position r is subject to a gravitational force

$$F = \left(\frac{mg}{R}\right) r$$

   directed towards the centre, where R is the radius of the earth and g the gravitational acceleration at the surface.

2. The force on the package is proportional to the distance from the centre. This is just like a spring! Let's understand springs first, then return to the delivery problem. If we attach a mass m to a spring of stiffness k, and pull the mass a distance x away from the equilibrium position, there is a restoring force

$$F = -kx.$$

[13] This explains why we always just treat planets as point masses in gravity problems.

If we displace the mass and let it go, the result is *simple harmonic motion*, where the mass just oscillates back and forth. To understand this motion, we can use the *phasor trick*. The basic idea is to upgrade x to a complex variable $\tilde{x} = re^{i\omega t}$ in uniform circular motion on the complex plane. Treating the acceleration $\tilde{a}$ and position $\tilde{x}$ as phasors, show that the phasor satisfies

$$\tilde{a} = -\omega^2 \tilde{x}.$$

Just so you know, you don't need any calculus!

3. Conclude that the phasor satisfies a spring equation for

$$k = \omega^2 m.$$

4. We must return to the harsh realities of the real line. To pluck out a real component of the phasor, we can use *Euler's formula*:

$$e^{i\omega t} = \cos(\omega t) + i \sin(\omega t).$$

By taking the real part of the phasor solution, show that a mass m oscillates on a spring of stiffness k according to

$$x(t) = x(0)\cos(\omega t), \quad \omega = \sqrt{\frac{k}{m}},$$

where it is released from rest at $x(0)$.

5. Using questions (1) and (4), argue that the package reaches the other side of the world in time

$$t_{\text{delivery}} = \pi\sqrt{\frac{R}{g}}.$$

Solution

1. Using the second part of the Sphere Theorem, a package at radius r only feels gravitational attraction to the mass within a sphere of radius r; the rest is a shell it feels no attraction. Moreover, by the first part of the Sphere Theorem, we can concentrate all the mass of the sphere at the centre, so the force is simply

$$F = \frac{GM(r)m}{r^2},$$

where $M(r)$ is the mass enclosed in the smaller sphere. If the mass of the earth is M, and it is constant density, then $M(r)$ is just

$$\frac{V(r)}{V(R)}M = \frac{r^3 M}{R^3}.$$

We also know that the acceleration at the surface is

$$g = \frac{GM}{R^2}.$$

It follows that

$$F = \frac{GM(r)m}{r^2} = \frac{GMmr}{R^3} = \left(\frac{mg}{R}\right) r.$$

2. Hopefully you remember that for uniform circular motion, the velocity v is related to the angular velocity ω and radius r by $v = \omega r$. The magnitude of acceleration is related to velocity by
$$|\tilde{a}| = \frac{v^2}{r} = \frac{\omega^2 r^2}{r} = \omega^2 r.$$
The acceleration is due to a centripetal force, so it is antiparallel to $\tilde{x}$. Thus, we discover that
$$\tilde{a} = -\omega^2 r e^{i\omega t} = -\omega^2 \tilde{x}.$$

3. If we assume a mass m is in uniform circular motion, our phasor result shows that
$$ma = -m\omega^2 x.$$
We can identify the RHS with the restoring force due to a spring, provided $k = m\omega^2$.

4. We have the phasor result
$$\tilde{x}(t) = re^{i\omega t}$$
for $\omega = \sqrt{k/m}$ and some fixed r we will interpret in a moment. Taking the real part gives
$$x(t) = \Re[\tilde{x}(t)] = r\Re[e^{i\omega t}] = r\cos(\omega t).$$
At $t = 0$, $x(0) = r$ is at its maximum extension (since $|\cos(\omega t)| \le 1$) and momentarily at rest. The real part therefore corresponds to releasing the mass at rest at r.[14] You may be worried that taking the real part spoils the relationship between a and x. It doesn't! The reason is simply that $\tilde{a} = -\omega^2\tilde{x}$ can be viewed as a *vector* statement, true for all components (i.e. the real and imaginary parts) as well as the vectors as a whole.

5. Going back to our result from (1), recall that we have something formally identical to a spring-mass system:
$$F = \left(\frac{gm}{R}\right) r = ma.$$
The corresponding angular frequency is
$$\omega = \sqrt{\frac{gm/R}{m}} = \sqrt{\frac{g}{R}}.$$
Now, a full period in the cosine $\cos(\omega t)$ occurs when
$$\omega(t+T) - \omega t = \omega T = 2\pi \quad \Longrightarrow \quad T = \frac{2\pi}{\omega}.$$
But the transit time for a package from one side of the earth to the other is precisely *half* a period. We finally learn the package delivery time is
$$t_{\text{delivery}} = \frac{1}{2}T = \frac{\pi}{\omega} = \pi\sqrt{\frac{R}{g}}.$$

[14] In general, we need to take some linear combination of sines and cosines, but that is a topic for another time!

3.4 Donuts and wobbly orbits

Take a square of unit length. By folding twice and gluing (see below), you can form a donut. Particles confined to the donut don't know it's curved; it looks like normal space to them, except that if they go too far to the left, they will reappear on the right, and similarly for the top and bottom. Put a different way, the blue lines to the left and right are identified, and similarly for the red lines. This is just like the video game *Portal*!

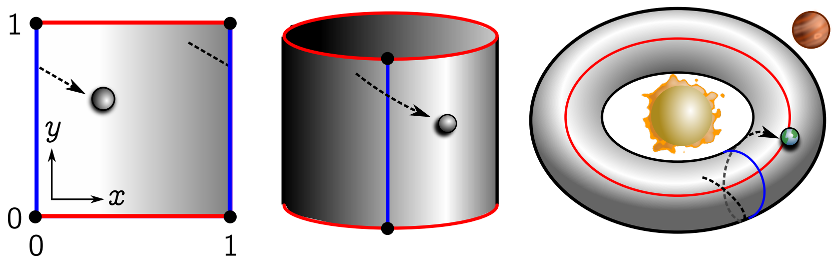


Figure 10: Folding and gluing a square to get a donut. The earth has a wobbly donut orbit (highly exaggerated) due to its attraction to Jupiter.

1. Suppose we have two particles, and shoot them out from the origin at $t = 0$. One particle travels vertically in the y direction with speed v_y, and the other travels in the x direction with speed v_x. Will they ever collide? If so, at what time will the first collision occur?

2. Now consider a *single* particle with velocity vector $\mathbf{v} = (v_x, v_y)$. Show that the particle will never visit the same location on the donut twice if the slope of its path cannot be written as a fraction of whole numbers. Such a non-repeating path is called *non-periodic*.

The earth orbits the sun, but feels a slight attraction to other planets, in particular the gas giant Jupiter. This attraction will deform the circular[15] orbit of the earth onto the surface of a donut, travelling like the particle in question (2). Sometimes, these small changes can accumulate over time until the planet flies off into space! This is obviously something we want to avoid. There is a deep mathematical result[16] which states that the orbit on the donut will be stable provided it is non-periodic. Periodic donut orbits, on the other hand, will reinforce themselves over time and create instabilities. This is like pushing a swing in sync with its natural rhythm: eventually, the occupant of the swing will fly off into space as well!

3. Regarding the x-direction as the circular direction around the sun, and y as the direction of the wobbling due to Jupiter, it turns out that

$$\frac{v_y}{v_x} = \frac{T_{\text{Jupiter}}}{T_{\text{earth}}}.$$

If the relative size of orbits is $R_{\text{Jupiter}} = 5R_{\text{Earth}}$, will the earth remain in a stable donut-shaped orbit? *Hint:* You may use the fact that $\sqrt{125}$ cannot be written as a fraction.

[15] In fact, the orbit is slightly stretched along one direction to form an ellipse, but we will ignore this point. One complication at a time!

[16] Called the *KAM theorem* after Kolmogorov, Arnol'd and Moser.

Solution

1. Since the first particle travels on the red line (y-axis) and the second particle travels on the blue line (x-axis), they will only collide if they both return to the origin at the same time. But this means that both must travel an *integer* distance in the same time, so for some natural numbers m_x, m_y, and some time t,

$$v_x t = m_x, \quad v_y t = m_y.$$

Dividing one equation by the other, we find that the ratio of velocities must be a fraction:

$$\frac{v_x}{v_y} = \frac{m_x}{m_y}.$$

If m_x, m_y have no common denominators, then the first time the particles coincide for $t > 0$ is when $v_x t = m_x$ and $v_y t = m_y$, so $t = v_x/m_x = v_y/m_y$. If the ratio of velocities is not a fraction, they can never collide.

2. This is just the first problem in disguise! The two particles get associated to the x and y coordinates of the single particle. To begin with, suppose the particle starts at the origin at $t = 0$. Let's look for conditions which stop it from returning there. From the first problem, it will never return to the origin as long as v_x/v_y is *irrational*. But there is nothing special about the origin; the same reasoning shows that if the ratio of velocity components is irrational, it will never return to any position it occupies.[17]

3. Kepler's third law states that the radius of an orbit R and the period T (i.e. the length of the year on the planet) are related by

$$T^2 = \alpha R^3$$

for some constant α which is the same for all planets. Thus,

$$\frac{T_{\text{Jupiter}}}{T_{\text{earth}}} = \frac{\sqrt{\alpha} R_{\text{Jupiter}}^{3/2}}{\sqrt{\alpha} R_{\text{earth}}^{3/2}} = 5^{3/2} = \sqrt{125}.$$

Since this cannot be expressed as a fraction, the results of part (2) show that the orbit is non-periodic. This means that the earth should stay in a stable donut orbit forever![18]

[17] Something even more remarkable happens: the one-dimensional trajectory of the particle manages to fill in most of the the two-dimensional surface of the donut! (It visits everywhere except a miniscule subset of area zero.)

[18] In fact, Jupiter's orbit is only approximately five times larger. But it remains true that a Jupiter year is some irrational number of earth years, which is the key to the stability of the earth's orbit.

4 Black holes

4.1 Colliding black holes and LIGO

When a star runs out of nuclear fuel, it can collapse under its own weight to form a black hole: a region where gravity is so strong that even light is trapped. Black holes were predicted in 1915, but it took until 2015, 100 years later, for the Laser Interferometer Gravitational-wave Observatory (LIGO) to observe them directly. When two black holes collide, they emit a characteristic "chirp" of *gravitational waves* (loosely speaking, ripples in spacetime), and through an extraordinary combination of precision physics and engineering, LIGO was able to hear this chirp billions of light years away.

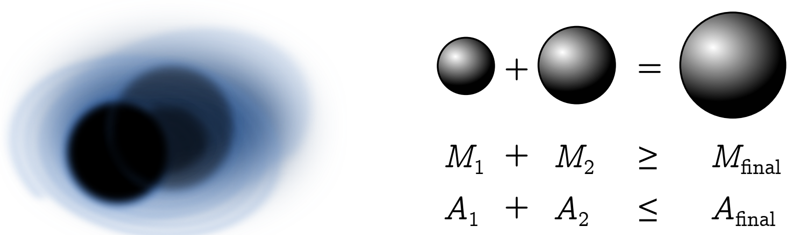


Figure 11: On the left, a cartoon of a black hole merger. On the right, inequalities obeyed by mergers: the mass of the final black hole can decrease when energy is lost (e.g. to gravitational waves), but the area always increases.

1. An infinitely dense point particle of mass M will be shrouded by a black hole. Using dimensional analysis, argue that this black hole has surface area

$$A = \left(\frac{\eta G^2}{c^4}\right) M^2$$

   for some dimensionless constant η.

2. One of Stephen Hawking's famous discoveries is the *area theorem*: the total surface area of any system of black holes increases with time.[19] Using the area theorem, and the result of part (1), show that two colliding black holes can lose at most 29% of their energy to gravitational waves. (Note that to find this upper bound, you need to consider varying the mass of the colliding black holes, and to assume that any lost mass is converted into gravitational waves.)

3. LIGO detected a signal from two black holes smashing into each other 1.5 billion light years away. Their masses were $M_1 = 30M_\odot$ and $M_2 = 35M_\odot$, where $M_\odot \approx 2 \times 10^{30}\,\mathrm{kg}$ is the mass of the sun, and the signal lasted for 0.2 seconds. Assuming the maximum amount of energy is converted into gravitational waves, calculate the average power

[19]This theorem is actually violated by quantum mechanics, but for large black holes, the violations are small enough to be ignored.

$P_{\rm BH}$ emitted during the collision. Compare this to the power output of all the stars in the universe, $P_{\rm stars} \sim 10^{49}\,\mathrm{W}$.

Solution

1. A black hole, by definition, is a gravitational trap for light. It will therefore involve Newton's constant G, which is related to the strength of gravity, and the speed of light c. The mass of the particle is also relevant, since we expect a heavier particle to correspond to a larger black hole. We denote the units of a quantity by square brackets, $[\cdot]$. Obviously, $[M] = \text{mass}$ and $[c] = \text{distance}/\text{time}$. From Newton's law of gravitation,

$$F = \frac{GMm}{r^2} \quad\implies\quad [G] = \frac{[F][r]^2}{[M]^2} = \frac{\text{length}^3}{\text{time}^2 \cdot \text{mass}},$$

where we used

$$[F] = [ma] = \text{mass} \cdot \frac{\text{length}}{\text{time}^2}.$$

Area has the units of length^2. We can systematically analyse the units using simultaneous equations, but here is a shortcut: time doesn't appear in the final answer, so we must combine G and c as G/c^2, which has units

$$[Gc^{-2}] = \frac{\text{length}}{\text{mass}}.$$

To get something with units length^2, we must square this and multiply by M^2. It follows that, up to some dimensionless constant η, the area of the black hole is

$$A = \left(\frac{\eta G^2}{c^4}\right) M^2.$$

2. Consider two black holes of mass M_1, M_2. The initial and final area are

$$A_{\rm init} = A_1 + A_2 = \frac{\eta G^2}{c^4}(M_1^2 + M_2^2), \quad A_{\rm final} = \left(\frac{\eta G^2}{c^4}\right) M_{\rm final}^2.$$

If $A_{\rm init} = A_{\rm final}$, we have maximal loss of mass; if $M_{\rm final} = M_1 + M_2$, we minimise the mass loss. The percentage of mass lost will depend on the mass of the black holes, but to place an upper bound, we want to choose the masses to maximise the fraction of mass lost. The simplest way to proceed is to instead look at the difference of *squared* masses,

$$\Delta M^2 = M_{\rm final}^2 - M_1^2 - M_2^2 = (M_1 + M_2^2)^2 - M_1^2 - M_2^2 = 2M_1 M_2.$$

Since we only care about the fraction lost, we can require a total initial mass $M = M_1 + M_2$ for fixed M, and now try to choose M_1, M_2 to maximise the square of mass lost:

$$\Delta M^2 = 2M_1 M_2 = 2M_1(M - M_1).$$

This is just a quadratic in M_1, with roots at $M_1 = 0$ and $M_1 = M$. The *maximum* will be precisely in between, at $M_1 = M/2$. Of course, maximising the square of lost mass should be the same as maximising the lost mass itself, so we obtain an upper bound on mass loss in any black hole collision by setting $M_1 = M_2$, with a fractional loss

$$1 - \frac{M_{\text{final}}}{M_1 + M_2} = 1 - \frac{\sqrt{M_1^2 + M_1^2}}{M_1 + M_1} = 1 - \frac{\sqrt{2}}{2} \approx 0.29.$$

Since the mass can be converted into gravitational waves, we have the 29% bound we were looking for!

3. From the last question, we know that we maximise the energy converted into gravitational waves when the total area doesn't change,

$$A_{\text{final}} = A_1 + A_2 = \frac{\eta G^2}{c^4}(M_1^2 + M_2^2) = \left(\frac{\eta G^2}{c^4}\right) M_{\text{final}}^2.$$

This corresponds to a loss of mass

$$\Delta M = M_1 + M_2 - M_{\text{final}} = M_1 + M_2 - \sqrt{M_1^2 + M_2^2} \approx 18.9\, M_\odot.$$

We can convert this to energy using the most famous formula in physics, $E = mc^2$. To find the average power P, we divide by the duration of the signal $t = 0.2\,\text{s}$. We find

$$P_{\text{BH}} = \frac{E}{t} = \frac{\Delta M c^2}{t} = \frac{18.9 \cdot 2 \cdot 10^{30}(3 \times 10^8)^2}{0.2}\,\text{W} \approx 1.7 \times 10^{49}\,\text{W}.$$

Since $P_{\text{BH}} > P_{\text{stars}}$, we see that for a brief moment, colliding black holes can outshine all the stars in the universe![20]

References

- "Gravitational radiation from colliding black holes" (1971). Stephen Hawking.

[20]This suggests that black hole mergers should be easy to see, but the analogy to starlight is misleading. Light likes to interact with things and can be easily absorbed, e.g. by the rods and cones in your eye, or the CCDs in a digital camera. In contrast, gravitational waves simply pass through matter, wobbling things a little as they go by. This wobbling is very subtle; so subtle, in fact, that an isolated observer can never detect it! But if we *very carefully* compare the paths of two photons going in different directions, we can discern the wobbling. This is why LIGO has two giant arms at right angles: one for each photon path.

4.2 Einstein rings

According to *general relativity*, Einstein's theory of gravity, massive objects curve space itself. Even *massless* particles like light rays will be deflected as they try to find the shortest path between A and B. This effect is called *gravitational lensing*, since a heavy body acts like a lens.

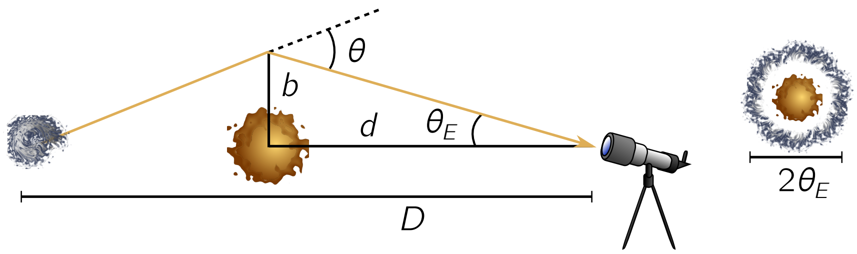


Figure 12: Einstein ring, gravitationally lensed by a large star.

1. Suppose that a light ray passes a spherical star of mass m and radius R, a distance $b > R$ from the centre. The *angle of deflection* θ is dimensionless (in radians). Using dimensional analysis, argue that it takes the form

$$\theta = c_0 + c_1 x + c_2 x^2 + \cdots$$

for dimensionless constants $c_0, c_1, c_2, \ldots$, and

$$x = \frac{Gm}{bc^2}.$$

The speed of light is $c = 3 \times 10^8$ m/s and Newton's constant is $G = 6.7 \times 10^{-11}$ m^3/kg s^2.

Hint. You may assume the wavelength of light isn't relevant, since only the mass $m = 0$ determines its path in spacetime. Why isn't the radius of the star relevant?

2. By considering the limit where the star disappears altogether, $m \to 0$, explain why $c_0 = 0$.

3. Using parts (1) and (2), argue that for $Gm \ll bc^2$,

$$\theta \sim \frac{Gm}{bc^2}.$$

As usual, the $\sim$ includes the unknown constant c_1.

Imagine that a star lies directly between a galaxy and a telescope on earth.The galaxy is a distance D away from the earth, and the star a distance d. Define the angle θ_E and deflection angle θ as in Fig. 4.2.

4. Assuming the angles are small, argue that

$$b \approx \theta_E d, \quad \theta_E D \approx \theta(D - d).$$

5. Combining the identities in (4) with (3), deduce that

$$\theta_E \sim \sqrt{\frac{Gm(D-d)}{c^2 Dd}}.$$

You can repeat this argument, rotating in a circle around the line formed by the galaxy, star and observer on earth. We learn that the galaxy will appear as a ring, called an *Einstein ring*, of (angular) *Einstein radius* θ_E.

6. Explain why we don't observe Einstein rings around the sun. The sun has mass $m_\odot = 2 \times 10^{30}\,\mathrm{kg}$, radius $R_\odot = 7 \times 10^8\,\mathrm{m}$, and is $d = 150 \times 10^9\ \mathrm{m}$ from earth.

Solution

1. The radius of the star is irrelevant due to the *sphere theorem*: bodies outside a uniform sphere of mass m are gravitationally attracted to the sphere as if all the mass were concentrated at the centre.[21] We are also told that the wavelength of light is irrelevant. This leaves a few important parameters:
   - the mass m of the star, with dimension of mass $[m] = M$;
   - the distance b at which the light ray is deflected, dimension length $[b] = L$;
   - the speed of light c, dimensions $[c] = L/T$;
   - and finally Newton's constant G, governing the strength of gravity, which has dimensions $[G] = L^3/MT^2$ from the units.

   There is a unique way to combine these to get a dimensionless constant:

   $$x = \frac{Gm}{bc^2},$$

   as you can easily check.[22] The deflection angle θ is dimensionless, provided we use radians. Since x can be raised to any power and still be dimensionless,[23] we have to write our dimensional guess as a sum of all these possibilities:

   $$\theta \sim c_0 + c_1 + c_2 x^2 + \cdots$$

2. When the star disappears, $m \to 0$, and there should be no deflection at all: $\theta = 0$. But when m disappears, all the positive powers of x vanish, and we are left with $\theta = c_0$. So we conclude that $c_0 = 0$.

3. By definition, $Gm \ll bc^2$ means

   $$x = \frac{Gm}{bc^2} \ll 1.$$

[21] In general relativity, the corresponding statement is known as *Birkhoff's theorem*.

[22] A simple argument is that the only way to cancel the units of time is to have G/c^2; to cancel units of mass we need Gm/c^2; and finally, to cancel units of length we take Gm/bc^2.

[23] You might wonder we why don't add powers of $x^{-1} = bc^2/Gm$. The simple answer is that they predict an *infinite* answer when the mass is small, which as we will see in the next question, is not the case.

It follows that

$$\theta = c_1 x + c_2 x^2 + \cdots \approx c_1 x,$$

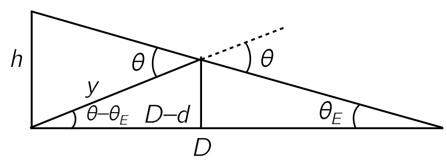
since higher powers of x are much smaller. Ignoring the constant c_1,[24] we have the dimensional analysis result that

$$\theta \sim \frac{Gm}{bc^2}.$$

4. From the diagram, we have $\tan\theta_E = b/d$. But for small angles, $\tan\theta_E \approx \theta_E$, so we find that

$$b \approx \theta_E d.$$

From the diagram, $h \approx \theta_E D \approx \theta y$ for small angles.



But for $\theta, \theta_E \ll 1$, the small angle approximation gives

$$\frac{D-d}{y} = \cos(\theta - \theta_E) \approx 1 \quad \Longrightarrow \quad y \approx D - d.$$

So we deduce that $\theta_E D \approx \theta(D-d)$.

5. Combining our previous results, we obtain

$$\theta_E \frac{D}{D-d} \approx \theta \sim \frac{Gm}{bc^2} \approx \frac{Gm}{\theta_E d c^2}.$$

Rearranging to make θ_E the subject, and taking a square root, we find as required that

$$\theta_E \sim \sqrt{\frac{Gm(D-d)}{c^2 Dd}}.$$

6. To see an Einstein ring, we require the size of the ring image at distance d to be larger than the radius R of the star, or $\theta_E d > R$. But

$$\theta_E d \sim \sqrt{\frac{Gmd(D-d)}{c^2 D}}.$$

The expression involving the distance looks tricky, but assumes a maximum of d:

$$\frac{d(D-d)}{D} < \frac{dD}{D} = d.$$

Thus, the condition to see Einstein rings from a source at *any* distance is $Gmd/c^2R^2 > 1$. We can plug numbers into this ratio and check its value for the sun:

$$\frac{Gmd}{R^2c^2} = \frac{(6.7\times 10^{-11})(2\times 10^{30})(150\times 10^9)}{(3\times 10^8)^2(7\times 10^8)^2} \approx 0.0005.$$

Since this is much less than 1, we have no hope of seeing solar Einstein rings!

[24] Different theories of gravity make different predictions for c_1: Newtonian gravity predicts $c_1 = 2$, while Einstein's general relativity predicts $c_1 = 4$. In 1919, Arthur Eddington was able to precisely check the deflection of starlight during a solar eclipse, and found that Einstein was correct!

4.3 Black hole hard drives

Black holes are perhaps the most mysterious objects in the universe. For one, things fall in and never come out again. An apparently featureless black hole could conceal an elephant, the works of Shakespeare, or even another universe! Suppose we wanted to describe all the possible objects that could have fallen into the black hole, but using *binary digits* (bits) 0 and 1, the language of computers. With one bit, we can describe two things, corresponding to 0 and 1; with two bits, we can describe *four* things, corresponding to $00, 01, 10, 11$. Continuing this pattern, with n bits we can describe 2^n things, corresponding to the 2^n sequences of n binary digits. The total number of bits needed to describe all the possibilities, for a given black hole, is called the *entropy* S. Since information is also stored in bits, we can (loosely) equate entropy and information!

We would expect that a large black hole can conceal more than a small black hole, and will therefore have a larger entropy. The *area law*, discovered by Stephen Hawking and Jacob Bekenstein, shows that this is true, with the entropy of the black hole proportional to its *surface area* A:

$$S = \frac{A}{A_0},$$

where $A_0 \approx 10^{-69}\,\mathrm{m}^2$ is a basic unit of area. We can view the black hole surface as a sort of screen, made up of binary pixels of area A_0.

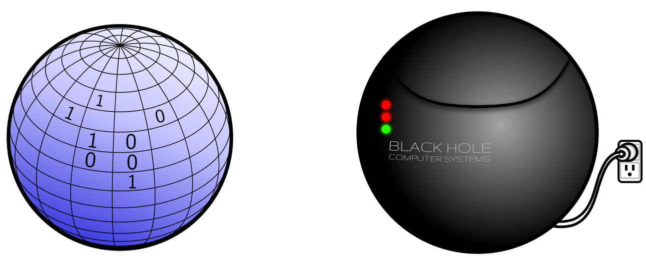


Figure 13: *Left*. The area law, viewed as pixels on the black hole surface. *Right*. A spherical hard drive.

The *Second Law of Thermodynamics* states that the total entropy of a closed system always increases.[25] Combining the area law and the Second Law leads to a surprising conclusion: black holes have the *highest* entropy density of any object in the universe. They are the best hard drives around![26]

1. To get a sense of scale, calculate how many gigabytes of entropy can be stored in a black hole the size of a proton, radius $\sim 10^{-15}\,\mathrm{m}$. Note that

$$1\,\mathrm{GB} = 10^9\,\mathrm{B} = 8 \times 10^9\,\mathrm{bits}.$$

[25] The entropy of a black hole is the number of bits needed to describe all the things that could have fallen in. The entropy of an ordinary object, like a box of gas, is the number of bits needed to describe all the different *microscopic* configurations which are indistinguishable to a macroscopic experimentalist, i.e. which look like the same box of gas. The function of entropy, in both cases, is to count the number of configurations which look the same!

[26] At least when it comes to information storage density. *Extracting* information is much harder!

Compare this to the total data storage on all the computers in the world, which is approximately $1.5 \times 10^{12}\,\mathrm{GB}$.

2. Consider a sphere of ordinary matter of surface area A. Suppose this sphere has more entropy than a black hole,
$$S' > S_{\mathrm{BH}} = \frac{A}{A_0}.$$
Argue that this violates the Second Law. You may assume that as soon as a system of area A reaches the mass M_A of the corresponding black hole, it immediately collapses to form said black hole. *Hint.* How could you force it to collapse?

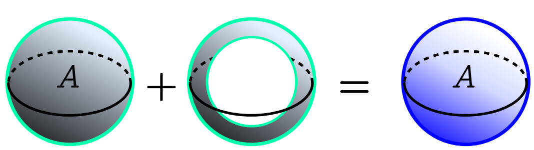
3. Calculate the optimal information density in a spherical hard drive of radius r.

4. Suppose that the speed at which operations can be performed in a hard drive is proportional to the density of information storage. (This is reasonable, since data which is spread out takes more time to bring together for computations.) Explain why huge (spherical) computers are necessarily slow.

Solution

1. We calculate the entropy from the area law, and convert the answer from bits to GB, then to multiples of the world's total storage:
$$\begin{aligned} S &= \frac{A}{A_0}\,\mathrm{bits} \\ &\approx \frac{4\pi(10^{-15})^2\,\mathrm{m}^2}{10^{-69}\,\mathrm{m}^2}\,\mathrm{bits} \\ &\approx 1.25 \times 10^{40}\,\mathrm{bits} \\ &\approx \frac{1.25 \times 10^{40}}{8 \times 10^9}\,\mathrm{GB} \\ &\approx 1.6 \times 10^{30}\,\mathrm{GB} \approx 10^{18}\,\text{global computer storage}. \end{aligned}$$
A proton-sized black hole contains more information than all the world's computers, by an unimaginably large factor $\sim 10^{18}$. That's roughly the number of grains of sand in the world! Perhaps GoogleX is working on black hole hard drives as we speak.

2. First, note that the mass of the sphere M must be smaller than the mass of the corresponding black hole M_A, otherwise it would have already collapsed! We can therefore add a spherical shell of matter, mass $M_A - M$, and compress it to ensure the surface area is A. By assumption, this spherical object will immediately collapse to form a new black hole. Schematically, we are performing the following "sum":



The shell of matter has its own entropy S'', so the total entropy of system before collapse is larger than the black hole entropy:

$$S' + S'' > S' > S_{\mathrm{BH}}.$$

However, after the collapse, the entropy is just the black hole entropy S_{BH}. So we seem to have reduced the total entropy! This violates the Second Law of Thermodynamics. Our assumption, that $S' > S_{\mathrm{BH}}$, must have been incorrect. We learn that black holes are the best spherical hard drives in existence!

3. Black holes have maximum entropy density. Using the area law, the entropy density of a black hole of radius r is
$$\frac{S}{V} = \frac{4\pi r^2}{A_0 4\pi r^3/3} = \frac{3}{A_0 r}.$$

4. The previous result shows that, as a spherical hard drive gets large, the *maximum* information density gets very low. Since this is a maximum, density and hence processing speed is low in *any* large hard drive.

5 Particle physics

5.1 Evil subatomic twins

In 1928, Paul Dirac made a startling prediction: the electron has an evil twin, the *anti-electron* or *positron*. The positron is the same as the electron in every way except that it has positive charge $q = +e$, rather than negative charge $q = -e$. In fact, *every* fundamental particle has an evil, charge-flipped twin; the evil twins are collectively called *antimatter*.[27]

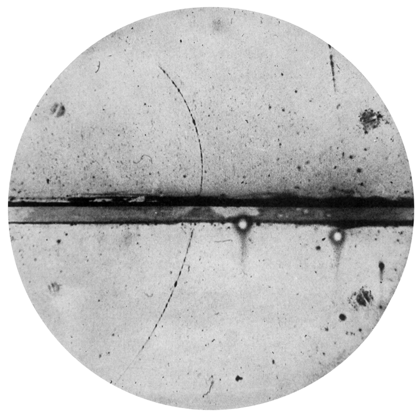


Figure 14: The mysterious trail in Carl Anderson's cloud chamber.

Experimentalist Carl Anderson was able to verify Dirac's prediction using a *cloud chamber*,[28] a vessel filled with alcohol vapour which is visibly ionised when charged particles (usually arriving from space) pass through it. In August 1932, Anderson observed the mysterious track shown above. Your job is to work out what left it!

1. A magnetic field $B = 1.7\,\mathrm{T}$ points into the page in the image above. Suppose that a particle of charge q and mass m moves in the plane of the picture with velocity v. Show that it will move in a circle of radius $R = mv/Bq$, and relate the sign of the charge to the motion.

2. The thick line in the middle of the photograph is a lead plate, and particles colliding with it will slow down. Using this fact, along with part (1), explain why the track in the image above must be due to a positively charged particle.

3. The width of the ionisation trail depends on what type of particle is travelling through the chamber and how fast it goes. The amount of ionisation in the picture above is consistent with an electron, but also an energetic proton, with momentum

$$p_{\mathrm{p}} \sim 10^{-16}\,\frac{\mathrm{kg}\cdot\mathrm{m}}{\mathrm{s}}.$$

[27]You may think it is a unfair to call antimatter "evil", but if you met your antimatter twin, hugging them would be extremely deadly! You would annihilate each other, releasing the same amount of energy as a large nuclear bomb.

[28]Cloud chambers are the modest ancestor of particle physics juggernauts like the Large Hadron Collider (LHC). Unlike the LHC, you can build a cloud chamber in your backyard!

Can you rule the proton out?

Solution

1. The Lorentz force law tells us that the particle is subject to a constant force of magnitude $F = Bqv > 0$. The force will be normal to the direction of motion, acting centripetally and causing the particle to move in a circle. To find the radius, we use $a = v^2/R$:

$$a = \frac{F}{m} = \frac{Bqv}{m} = \frac{v^2}{R} \quad \Longrightarrow \quad R = \frac{mv}{Bq}.$$

   Finally, by the right-hand rule, a positively charged particle will experience a force to its left, causing it to move around the circle anticlockwise (seen from above); similarly, a negatively charged particle will move clockwise.

2. From the previous question, the particle's radius of curvature will get smaller as it slows down. This tells us the particle in the image is moving from bottom to top. (Being able to tell which the particle is going is why Anderson added the plate!) Since its path curves in the anticlockwise sense, it must be positively charged.

3. The radius of the track is comparable to the radius of the chamber, $r \approx 0.1\,\mathrm{m}$. This leads to momentum

$$p = mv = BqR = 1.7 \times 0.1 \times (1.6 \times 10^{-19})\,\frac{\mathrm{kg}\cdot\mathrm{m}}{\mathrm{s}} \sim 10^{-20}\,\frac{\mathrm{kg}\cdot\mathrm{m}}{\mathrm{s}}.$$

   This is considerably smaller than the momentum a proton would need to create the trail seen in the photograph. This only leaves one option: it is the positron, the positively charged evil twin of the electron!

5.2 Quantum strings and vacuums

Suppose we stretch a string of length L between two fixed points. The string can oscillate sinusoidally in *harmonics*, the first few of which are sketched on the left below. Remarkably, by considering that harmonics of *space itself*, we can show that empty vacuum likes to push metal plates together!

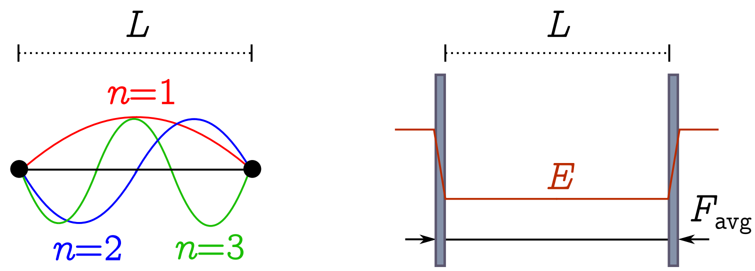


Figure 15: *Left.* Harmonics of a classical string. *Right.* Casimir effect on plates in a vacuum.

1. Show that harmonics on the string have wavelength

$$\lambda_n = \frac{2L}{n}, \quad n = 1, 2, 3, \ldots.$$

2. A classical string can vibrate with some combination of harmonics, including *no harmonics* when the string is at rest. In this case, the string has no energy. A *quantum* string is a little different: even if a harmonic is not active, there is an associated *zero-point energy*:

$$E_{0n} = \frac{\alpha}{\lambda_n},$$

where α is a constant of proportionality. This is related to *Heisenberg's uncertainty principle*, which states that we cannot know both the position and momentum of the string with absolute certainty. Let's calculate the zero-point energy of a quantum string.

Sum up the zero-point energies for each harmonic to find the energy of an unexcited quantum string. Use the infamous result[29] that

$$1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}.$$

3. Classical strings can be found everywhere, but where do we find quantum strings? One answer is *space itself*. Instead of stretching a string between anchors, set two lead plates a distance L apart. (Pretend that it vibrates in a plane, as in the picture above.) The harmonics are no longer wobbling modes of the string, but *electromagnetic waves*. Outside the plates is empty space, stretching away infinitely; it has zero energy.[30]

[29] There are various ways of showing this, but the basic idea is that very large numbers in this sum correspond to high frequencies which would break the string if we tried to excite them. So we have to throw most of these large numbers away, i.e. subtract them from our running tally. In the process, we overcorrect and get a slightly negative result!

[30] We can model the edge of space with lead plates infinitely far away. Since $L \to \infty$, $E_n^0 \to 0$ and the energy does indeed disappear.

Suppose that the lead plates have thickness ℓ. Show that the plates are pushed together, with each subject to an average force

$$F_{\text{avg}} = \frac{\alpha}{24\ell L}.$$

The remarkable fact that the vacuum can exert pressure on parallel metal plates is called the *Casimir effect*. Although weak, it can be experimentally detected!

Bonus. These methods can also be applied to *string theory*. String theory posits that everything in the universe is made out of tiny vibrating strings. Different subatomic particles, like electrons and photons, correspond to the different ways that the string can vibrate. We will learn that string theory requires 25 spatial dimensions![31]

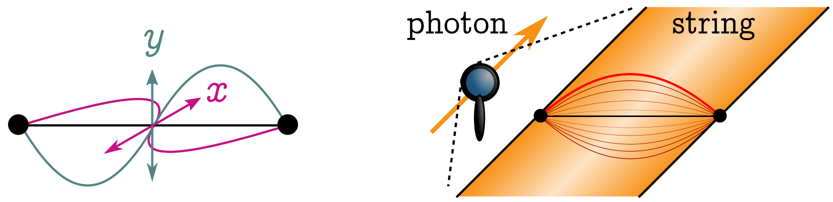


Figure 16: *Left*. Strings vibrating in different independent directions. *Right*. If we zoom in on a photon, we get a string with a single excited harmonic.

When we treat the string as a quantum object, each independent direction gets independent harmonics. Put a different way, we can split the string into $D-1$ independent strings wobbling in two dimensions, labelled by $i = 1, 2, \ldots, D-1$. To get different fundamental particles, we need to be able to *excite* harmonics. It turns out that, according to quantum theory, they have discrete *energy levels*, separated by "quantum leaps" in energy:

$$E^i_{mn} = \frac{\alpha}{\lambda_n}(1 + 2m), \quad m = 0, 1, 2, \ldots.$$

The superscript i denotes the direction the harmonic wobbles; the subscript n refers to the harmonic, while m refers to how excited that harmonic is. To find the total energy of the string, we just add up the energy of each harmonic.

4. The string can vibrate in any direction perpendicular to the string. In three spatial dimensions, there are two perpendicular directions for the string to vibrate (labelled by x and y above). Explain why, for D spatial dimensions, the string can vibrate in $D-1$ independent directions.

5. Suppose that we excite a first harmonic ($n = 1$) in some direction to its lowest excited state ($m = 1$). A string vibrating this way looks like a *photon* from far away, i.e. a particle of light. Use the fact that the photon *has zero mass* to deduce that $D = 25$.

[31] Since we only see three dimensions, the remaining 22 must somehow be "curled up" and hidden from view.

Solution

1. From the picture, we see that λ is an allowed wavelength if L is a multiple of $\lambda/2$. More precisely,
$$L = \frac{n\lambda}{2} \implies \lambda_n = \frac{2L}{n}.$$
2. The total rest energy of the quantum string is
$$E^0 = \frac{\alpha}{\lambda_1} + \frac{\alpha}{\lambda_2} + \frac{\alpha}{\lambda_3} + \cdots = \frac{\alpha}{2L}(1 + 2 + 3 + \cdots) = -\frac{\alpha}{2L} \cdot \frac{1}{12} = -\frac{\alpha}{24L}.$$
3. A jump in energy ΔE in energy over a distance Δx leads to an average force
$$F_{\text{avg}} = -\frac{\Delta E}{\Delta x}.$$
In this case, the distance over which the energy drops is the thickness of the plates, $\Delta x = \ell$, while the change in energy (as we move into the area between plates) is
$$\Delta E = E_{\text{plates}} - E_{\text{vacuum}} = E_{\text{plates}} = \frac{\alpha}{24L},$$
since the energy for the electromagnetic waves between plates takes the same form as harmonics in the stretched string. Thus, the average force on each plate is
$$F_{\text{avg}} = -\frac{E^0}{\ell} = \frac{\alpha}{24\ell L}.$$
This is positive, hence directed *towards* the region between plates. This means the plates are squeezed together!

4. If there are D directions, then one direction is parallel to the string, and the remaining $D - 1$ directions are perpendicular to it. Thus, there are $D - 1$ independent directions the string can wobble in.

5. There are $D - 2$ directions with all harmonics at rest, and one direction with its first harmonic (the red vibration in the picture above) in its first energy level. From question 2, the unexcited directions have total rest energy
$$E_0 = -\frac{\alpha}{24L}.$$
From the expression for E^i_{mn}, we see that by setting $m = n = 1$, we add an energy
$$\frac{2\alpha m}{\lambda_n} = \frac{2\alpha}{\lambda_1} = \frac{\alpha}{L}$$
to the unexcited energy of the harmonic. Thus, the total energy of the string is
$$E = (D-2)E_0 + \left(E_0 + \frac{\alpha}{L}\right) = \frac{\alpha}{L}\left(-\frac{D-1}{24} + 1\right).$$
If the photon is massless, then $m = 0$, and by the most famous formula in physics, $E = mc^2 = 0$. This implies that
$$-\frac{D-1}{24} + 1 = 0 \implies D = 25.$$
If string theory is correct, and photons have no mass, then the universe has 25 dimensions!