UBC Virtual Physics Circle
A Hacker's Guide to Brownian motion

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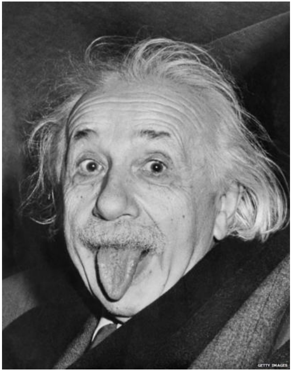
Overview

- Our last topic in the hacker's guide is Brownian motion. It's the other reason Einstein got a Nobel prize!

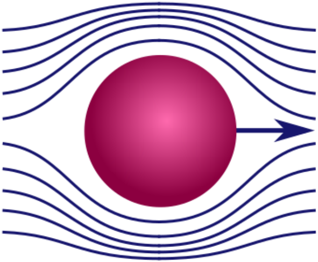


- We start with some review, then we do some thermodynamics. Add it all up to get Brownian motion!

Review

Review: Stokes' law

- First, we need to remind ourselves of a result from the lecture on dimensional analysis.
- A sphere of radius R moves at (slow) speed v through a fluid of viscosity η. (Units: $[\eta] = M/LT$.)



- Stokes' law states that the drag force is

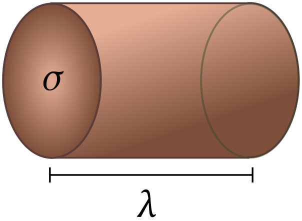
$$F_{\text{drag}} = 6\pi\eta v R.$$

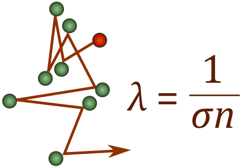
Review: random walks and collisions

- Last lecture, we introduced random walks and collisions.
- Walks: A random walk of N steps with length ℓ wanders

$$d \sim \sqrt{N}\ell.$$

- Collisions: If you have cross-section σ, and collide with stuff of density n (number per unit volume), your mean free path λ between collisions is





Speed and diffusion

- There is another useful piece of terminology.
- In some time t, suppose a walk spreads a distance d. The diffusion constant D is defined by

$$d = \sqrt{Dt}.$$

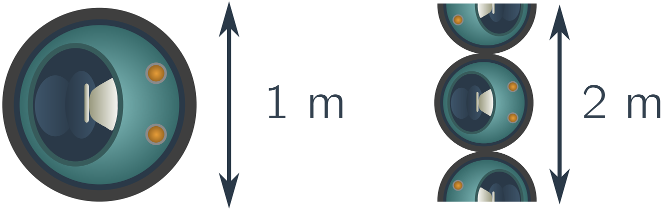
- Assume the walker moves at speed v. Each step takes time $\tau = \ell/v$, and N steps take time $t = N\tau$.Then

$$D = \frac{d^2}{t} = \frac{(\sqrt{N}\ell)^2}{t} = \frac{N\ell^2}{t} = \frac{\ell^2}{\tau} = v\ell.$$

- So the diffusion constant $D = v\ell$.

Exercise 1: dodgem cars

- Dodgem cars travel on average 1 m/s, with $\sigma \sim 2$ m.



- You and 4 of friends are colliding randomly on a square arena 5 m in width.
- 1. What is the mean free path λ?
  2. Roughly how long does it take to bounce from the center to the edge of the arena? Use $t = d^2/D = d^2/\ell v$.

Exercise 1: dodgem cars (solution)

- There are 5 cars in a 25 m^2 area, so $n = 0.2\ \text{m}^{-2}$. The cross-section is $\sigma = 2$ m, and hence the mfp is

$$\lambda = \frac{1}{\sigma n} = \frac{1}{2 \times 0.2}\ \text{m} = 2.5\ \text{m}.$$

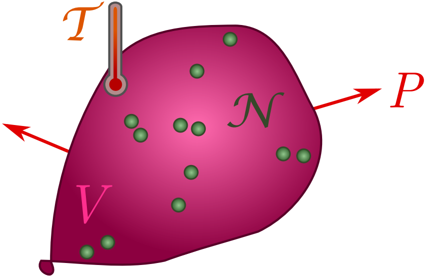
- Using the hint,

$$t = \frac{d^2}{\ell v} = \frac{25}{2.5 \times 1}\ \text{s} = 10\ \text{s}.$$

- In reality, you're trying to hit each other. So this is a good model only if, e.g. you fall asleep!

Review: ideal gas law

- To connect to atomic motion, we need to learn about the ideal gas law. (This may be review for many of you.)
- Imagine a balloon $\mathcal{N}$ gas particles.



- The gas is hot (temperature $\mathcal{T}$), takes up space (volume V), and presses on the balloon (pressure P).

Derivation of ideal gas law

- The ideal gas law states that these properties are related:

$$PV = k_B \mathcal{N} \mathcal{T}.$$

- Here, $k_B = 1.38 \times 10^{-23}$ J/K is Boltzmann's constant.
- We can "derive" this from dimensional analysis! But we need more than MLT (mass, length, time).
- In addition to length L and time T, use the following:
  1. energy $\mathcal{E}$ (instead of M);
  2. temperature Θ;
  3. and particle number Ξ.
- Ξ is for stuff growing with particle number, e.g. $[\mathcal{N}] = \Xi$.

Exercise 2: ideal gas

(a) Show that P and V have dimension

$$[V] = \Xi L^3, \quad [P] = \frac{\mathcal{E}}{L^3}.$$

(b) From $k_B = 1.38 \times 10^{-23}$ J/K, deduce $[k_B] = \mathcal{E}/\Theta$.

(c) Conclude that

$$[PV] = [\mathcal{N} k_B \mathcal{T}] = \Xi \mathcal{E}.$$

- With more care, you can show $PV = \mathcal{N} k_B \mathcal{T}$ is the only dimensionally consistent relation between all these things.

Exercise 2: ideal gas (solution)

(a) Usually, V has units $[V] = L^3$. But the volume of a gas grows with particle number, so $[V] = \Xi L^3$.

- As for pressure, using work ($Fd = W$)

$$P = \frac{F}{A} = \frac{W}{Ad} \quad \Longrightarrow \quad [P] = \frac{[W]}{[Ad]} = \frac{\mathcal{E}}{L^3}.$$

(b) We have

$$[k_B = 1.38 \times 10^{-23} \text{ J/K}] = [\text{J}]/[\text{K}] = \mathcal{E}/\Theta.$$

(c) From part (a), $[PV] = (\Xi L^3)(\mathcal{E}/L^3) = \Xi\mathcal{E}$.
From part (b), $[k_B \mathcal{N} \mathcal{T}] = (\mathcal{E}/\Theta)[\mathcal{N}][\mathcal{T}] = \Xi\mathcal{E}$.

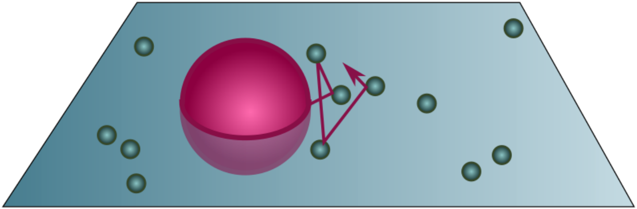
Brownian motion

Lucretius, Brown, Einstein

- A little history! In 60 BC, Roman philosopher Lucretius observed the zigzag motion of dust motes. He correctly attributed it to collisions with tiny invisible particles.
- In 1827, botanist Robert Brown saw pollen jiggle under a microscope. Unlike Lucretius, he couldn't explain it!
- Most 19th century scientists were skeptical of atoms.
- In 1905, a 26-year old Swiss patent clerk finished a PhD on Brownian motion, expanding on Lucretius' idea to account for the jiggling grains. That clerk: Einstein!

The pollen polka

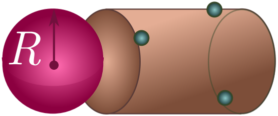
- We will reproduce one of the main results of Einstein's PhD thesis using cheap guesswork, i.e. hacking.
- Pour a viscous fluid into a container, then plonk a few spherical pollen grains into it, as below:



- The pollen (pink) will start jiggling around as it collides with fluid molecules (green), executing a random walk.

Brownian motion: mean free path

- Let's put it all together to see how far the pollen jitters. This is measured by the diffusion coefficient $D = \lambda v$.
- First, λ! The pollen is much larger than the molecules. If it has radius R, it has cross-section $\sigma = \pi R^2$.



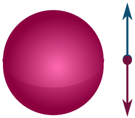
- Assume the fluid obeys the ideal gas law, $PV = k_B \mathcal{N} \mathcal{T}$. Since density $n = \mathcal{N}/V$, the mfp λ is

$$\lambda = \frac{1}{n\sigma} = \frac{V}{\mathcal{N}\pi R^2} = \frac{V}{\mathcal{N}\pi R^2} \times \frac{k_B \mathcal{N} \mathcal{T}}{PV} = \frac{k_B \mathcal{T}}{\pi P R^2}.$$

Brownian motion: terminal velocity

- What about the speed v? A reasonable guess is terminal velocity, achieved when weight balances drag force.
- From Stokes' law, if the fluid has viscosity η,

$$mg = 6\pi\eta v_{\text{term}} R \quad \Longrightarrow \quad v_{\text{term}} = \frac{mg}{6\pi\eta R}.$$



$$F_{\text{drag}} = 6\pi\eta v R$$

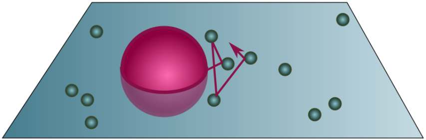
$$W = mg$$

- Combining with our expression for λ, we get

$$D = \lambda v_{\text{term}} = \frac{k_B T}{\pi P R^2} \cdot \frac{mg}{6\pi\eta R}.$$

Brownian motion: magic trick!

- A magic trick: suppose the pollen settles at a height where pressure balances weight, or $mg = PA = \pi P R^2$.



- Plugging this into D gives the Stokes-Einstein relation:

$$D = \frac{k_B \mathcal{T}}{\pi P R^2} \cdot \frac{mg}{6\pi\eta R} = \frac{k_B \mathcal{T}}{mg} \cdot \frac{mg}{6\pi\eta R} = \frac{k_B \mathcal{T}}{6\pi\eta R},$$

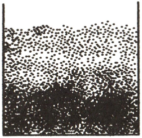
one of the main results of Einstein's PhD thesis!

Brownian motion: comments

- Thus, a pollen grain wanders a distance d in time t,

$$d \sim \sqrt{Dt}, \quad D = \frac{k_B T}{6\pi\eta R}.$$

- In 1908, Jean Perrin experimentally confirmed Einstein's predictions, hence the existence of atoms. Another Nobel!



- Grains don't settle at a fixed height, but exist in "dynamic equilibrium". We used a cheeky hacker shortcut!

Exercise 3: Avogadro's constant I

- You may have seen the chem version of the ideal gas law:

$$PV = \mathcal{N}_{\text{mol}}\mathcal{R}\mathcal{T},$$

where $\mathcal{R} = 8.3$ J/K mol is the ideal gas constant.

(a) Recall that one mol is N_A particles, where N_A is Avogadro's constant. Show that $N_A = \mathcal{R}/k_B$.

(b) In 1905, $\mathcal{R}$ was known but N_A was not. Argue that

$$N_A \sim \frac{t}{d^2} \cdot \frac{\mathcal{R}\mathcal{T}}{6\pi\eta R}.$$

This was one of no fewer than five methods Einstein proposed for measuring Avogadro's constant!

Exercise 3: Avogadro's constant I (solution)

(a) We equate the physics and chem version to get:

$$\begin{aligned} \mathcal{N} k_B \mathcal{T} &= PV = \mathcal{N}_{\text{mol}} \mathcal{R} \mathcal{T} \\ \implies \quad \mathcal{N} k_B &= \mathcal{N}_{\text{mol}} \mathcal{R}. \end{aligned}$$

Since $\mathcal{N}_{\text{mol}} N_A = \mathcal{N}$, we find

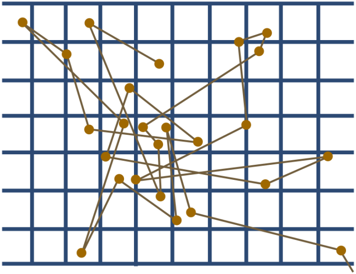
$$N_A k_B = \mathcal{R}.$$

(b) From the Stokes-Einstein relation,

$$\frac{k_B \mathcal{T}}{6\pi\eta R} \sim \frac{d^2}{t} \quad \implies \quad N_A \sim \frac{t}{d^2} \cdot \frac{\mathcal{R}\mathcal{T}}{6\pi\eta R}.$$

Exercise 3: Avogadro's constant II

- Below, we show some of Perrin's data ($R = 0.5\,\mu$m):



- Observations are made every 30 s, and lines ruled every $3\,\mu$m. The water had $T = 290$ K and $\eta = 0.011$ kg/m s.

(c) Using this data, find N_A.

Exercise 3: Avogadro's constant II (solution)

(c) We have 20 points, spread over 5 divisions or so.
Thus, $t = 30 \times 20$ s, and $d \sim 5 \times 3\,\mu$m.

- We have $\mathcal{T} = 290$ K, $R = 0.5\,\mu$ m, $\eta = 0.0011$ kg/m s. Plugging into (b) and using SI units everywhere,

$$\begin{aligned} N_A &\sim \frac{t}{d^2} \cdot \frac{\mathcal{R}\mathcal{T}}{6\pi\eta R} \\ &= \frac{30 \times 20}{(5 \times 3 \times 10^{-6})^2} \frac{8.3 \cdot 290}{6\pi \cdot 0.0011 \cdot (0.5 \times 10^{-6})} \\ &\approx 6.2 \times 10^{23}. \end{aligned}$$

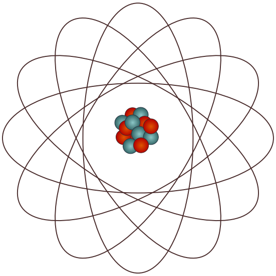
The modern value is $N_A = 6.022 \times 10^{23}$. Sweet!

Exercise 3: Avogadro's constant III

- Of course, the conventional definition of N_A is the number of carbon atoms in 12 g of carbon-12.

(d) From $N_A \approx 6 \times 10^{23}$, estimate a carbon-12 atom's mass.

- Most of the atom's mass is concentrated in its nucleus, made of (roughly) equally weighted protons and neutrons.



(e) What is the approximate mass of a nucleon?

Exercise 3: Avogadro's constant III (solution)

(d) The atom mass m_C is the total mass divided by N_A:

$$m_C = \frac{12\text{ g}}{N_A} \approx 2 \times 10^{-23}\text{ g} = 2 \times 10^{-26}\text{ kg}.$$

(e) We have $m_C \approx 12 m_{\text{nucleon}}$, and hence:

$$m_{\text{nucleon}} \approx \frac{m_C}{12} \approx 1.7 \times 10^{-27}\text{ kg}.$$

Questions?

Thanks everyone, you've been grand.

Go forth and hack physics!