Problems in Real Analysis

David Wakeham
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Curry's Paradox

We start with a simple identity due to philosopher Charles Peirce (1839–1914). An odd consequence is *Curry's paradox*, discovered by logician Haskell Curry (1900–1982). Like Russell's paradox (discussed in lectures), the paradox arises when we allow *self-reference*.

1. Show that Peirce's law,
$$((p \implies q) \implies p) \implies p$$
is a tautology via truth tables.

2. Suppose that $(p \implies q) \iff p$ is true. Use part (a) and modus ponens to deduce q.

3. Argue informally that, for any proposition q, the self-referential sentence
$$p = \text{“If } p \text{ is true, then it implies } q\text{”}$$
satisfies $(p \implies q) \iff p$.

4. Combine (b) and (c) to conclude that anything is true. What has gone wrong here?

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The Terrible Dynasties

Sets A and B are said to have the same *cardinality* if there exists a bijection (one-to-one, onto function) $f : A \to B$. Cardinality lets us think about the size of *infinite* sets.

1. For an infinite set X, consider a map $f : X \to \mathcal{P}(X)$. Show that f cannot be onto by considering the subset $R = \{x \in X : x \notin f(x)\}$. This means that sets are always "smaller" than their power sets. This result was proved by the founder of set theory, Georg Cantor (1845–1918). Hint: This is very similar to Russell's paradox.

2. Let $\aleph_0$ denote $|\mathbb{N}|$, the cardinality of the natural numbers. We call any cardinal of an infinite set an *infinite cardinal*; if you like, it is a "type of infinity". Let
$$\aleph_{n+1} \equiv |\mathcal{P}(A_n)|,$$
where A_n is a set with cardinality $\aleph_n$. Using part (a), argue that there is a tower of ever-bigger infinite cardinals
$$\aleph_0, \aleph_1, \aleph_2, \ldots$$
In other words, there is an infinite number of different infinities!

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Models and Non-implication

Suppose that we have a binary operation $\otimes$ ("bizarro" multiplication), which enjoys some combination of the following properties:

$$x \otimes (y \otimes z) = (x \otimes y) \otimes z \tag{A}$$
$$x \otimes x = x \tag{B}$$
$$(x \otimes y) \otimes z = x \otimes z. \tag{C}$$

If we want to show that some mathematical statements $A_1, A_2, \ldots, A_n$ (such as axioms) do not imply some other statement B, we need only find a single *model* of the situation where $A_1, A_2, \ldots, A_n$ are true but B is false. Truth tables are a special case of this, where we show a statement is *not* a tautology by finding a single assignment of truth values (a "model") which makes it false.

1. Show that if $x \otimes y = \max(x, y)$, then $\otimes$ satisfies (A) and (B) but not (C).

2. Find a binary operator which which satisfies (A) and (C) but not (B).

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Zippers and Hypercubes

Consider a real number in the unit interval, $x \in [0, 1]$. We can expand x as an infinite decimal

$$x = 0.d_1 d_2 d_3 \ldots, \quad d_i \in \{0, 1, \ldots, 9\}.$$

Thus, a real number between 0 and 1 can be represented as an infinite sequence of digits.

1. Are these digit sequence representations unique? If not, can we adopt conventions to make them unique?

2. Find a procedure to "smush" two digit sequences together to form a third sequence. Your procedure should be reversible, that is, you should be able to "un-smush" a digit sequence to uniquely recover the two digit sequences which were smushed to make it.

3. Use your answer to (b) to find a correspondence between the unit interval $[0, 1]$ and the unit square $[0, 1]^2 = [0, 1] \times [0, 1]$.

4. Extend the procedure from (b) to n digit sequences, and therefore deduce a correspondence between the unit interval [0, 1] and the n-cube

   $$[0, 1]^n = \overbrace{[0, 1] \times \cdots \times [0, 1]}^{n \text{ times}}.$$

   Remarkably, this shows that the unit interval is the *same size* (in the sense of set theory) as the unit hypercube in n dimensions!

Taming the Tails

A *summation machine* is an operator $\mathcal{S}$ which takes a sequence of real numbers and either (a) produces out a real number, or (b) gives up. We write the result of applying the machine to a sequence $\{a_1, a_2, a_3, \ldots\}$ as

$$\mathcal{S}\left[\sum_{n=1}^{\infty} a_n\right].$$

In the first case, we interpret the number it spits out as the result of adding all the numbers up, and say the series $\sum_n a_n$ *converges according to* $\mathcal{S}$. In the second, we say the series is *divergent* according to that procedure. In order to get a sensible addition operator $\mathcal{S}$, we impose two additional contraints:

$$\mathcal{S}\left[\sum_{n=1}^{\infty} a_n\right] = a_1 + \mathcal{S}\left[\sum_{n=2}^{\infty} a_n\right] \qquad \text{(additivity)}$$

$$\mathcal{S}\left[\alpha\sum_{n=1}^{\infty} a_n + \beta\sum_{n=1}^{\infty} b_n\right] = \alpha\mathcal{S}\left[\sum_{n=1}^{\infty} a_n\right] + \beta\mathcal{S}\left[\sum_{n=1}^{\infty} b_n\right]. \qquad \text{(linearity)}$$

1. Using additivity and linearity, show that if the following series converge according to $\mathcal{S}$, they must take specific values:

   (a) *Grandi's series:*

   $$\mathcal{S}[1 - 1 + 1 - 1 + \cdots] = \mathcal{S}\left[\sum_{n=0}^{\infty}(-1)^n\right] = \frac{1}{2}.$$

   (b) *Alternating natural numbers:*

   $$\mathcal{S}[1 - 2 + 3 - 4 + \cdots] = \mathcal{S}\left[\sum_{n=1}^{\infty}(-1)^n n\right] = \frac{1}{4}.$$

   Hint: Use (a) and additivity.

   (c) *Natural numbers:*

   $$\mathcal{S}[1 + 2 + 3 + 4 + \cdots] = \mathcal{S}\left[\sum_{n=1}^{\infty} n\right] = -\frac{1}{12}.$$

   Hint: Use (b), linearity, and $L - 4L = -3L$, where L is the limit.

2. The *Cesàro sum* (Ernesto Cesàro, 1859–1906) is the limit of the average of the first N partial sums:

   $$\mathcal{C}\left[\sum_{n=1}^{\infty} a_n\right] = \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\sum_{n=1}^{n} a_n = \frac{1}{N}\sum_{k=1}^{N} S_k.$$

   Check this is a summation machine, and verify that Grandi's series converges.

Simple Polylogarithms

We will investigate the series

$$a_m(n) = \sum_{k=1}^{\infty} \frac{k^n}{m^k}.$$

1. Show using an appropriate test that $a_m(n)$ converges for any $n \in \mathbb{N} \cup \{0\}$ and $|m| > 1$.

2. What is $a_m(0)$? Your answer will depend on m.

3. Show that

$$a_m(n) = \frac{1}{m} + \sum_{k=1}^{\infty} \frac{(k+1)^n}{m^{k+1}} = \frac{1}{m}\left[1 + \sum_{k=1}^{\infty} \frac{(k+1)^n}{m^k}\right].$$

4. Recall the *binomial theorem*

$$(k+1)^n = \sum_{j=0}^{n} \binom{n}{j} k^j.$$

   Using this identity, prove that

$$a_m(n) = \frac{1}{m-1}\left[1 + \sum_{j=0}^{n-1} \binom{n}{j} a_m(j)\right].$$

   HINT: You are allowed to swap the order of an infinite summation $\sum_{k=1}^{\infty}$ and a finite summation $\sum_{j=0}^{n}$.

5. We can calculate $a_m(0)$ using the results of (2). Using the identity in part (4), we can iteratively calculate any $a_m(n)$ we like! Put theory into practice, and explicitly evaluate the following series:

$$\sum_{k=1}^{\infty} \frac{1}{2^k}, \quad \sum_{k=1}^{\infty} \frac{k}{2^k}, \quad \sum_{k=1}^{\infty} \frac{k^2}{2^k}, \quad \sum_{k=1}^{\infty} \frac{k^3}{2^k}.$$

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Pi Makers

Power series and Taylor's theorem give us powerful methods for representing functions and constants. For instance, using the Taylor series for the tangent function (and Abel's theorem since we evaluated at an endpoint), we found that

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^n}{2n+1} + \cdots$$

In this problem, we prove a curious infinite product identity for π.

1. For $n = 0, 1, 2, \ldots$, let
$$I(n) = \int_0^\pi dx\, \sin^n(x).$$
Show that $I(0) = \pi$ and $I(1) = 2$. Using $\sin^2 x + \cos^2 x = 1$, and integration by parts, deduce that for $n \geq 2$,
$$I(n) = \frac{n-1}{n} I(n-2).$$

2. Use induction and (1) to prove that
$$I(2n) = \pi \prod_{k=1}^{n} \frac{2k-1}{2k}, \quad I(2n+1) = 2 \prod_{k=1}^{n} \frac{2k}{2k+1}.$$

3. By comparing integrands, show that $I(2n+1) \leq I(2n) \leq I(2n-1)$. Divide through by $I(2n+1)$ and use (1),
$$\lim_{n\to\infty} \frac{I(2n)}{I(2n+1)} = 1.$$

4. Rewriting the limit in (3), obtain the final result:
$$\frac{\pi}{2} = \lim_{n\to\infty} \prod_{k=1}^{n} \left(\frac{2k}{2k-1} \cdot \frac{2k}{2k+1} \right) = \left(\frac{2}{1} \cdot \frac{2}{3} \right) \left(\frac{4}{3} \cdot \frac{4}{5} \right) \left(\frac{6}{5} \cdot \frac{6}{7} \right) \cdots$$

Fourier from Power Series

Adding *negative* powers x^{-n} to a power series yields what is called a *Laurent series*. These converge on an *annulus* rather than a disc. Laurent series are important in complex analysis, where instead of real x, we have a function F of a complex variable $z \in \mathbb{C}$:

$$F(z) = \sum_{k \in \mathbb{Z}} a_k z^k.$$

We can use these to derive Fourier series.

1. We can restrict z to the unit circle in $\mathbb{C}$ via $z = e^{i\theta}$. Let $f(\theta) = F\left(e^{i\theta}\right)$. Argue that the function f is periodic with period 2π, and give a Laurent series for $f(\theta)$.

2. Integrate $f(\theta)e^{-i\ell\theta}$ for $\ell \in \mathbb{Z}$, $\theta \in [0, 2\pi)$. Use this to give an integral expression for a_k in terms of $f(\theta)$.

   HINT: You may interchange integration and summation.

3. Suppose that $f(\theta)$ is real. Using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, and writing $a_k = b_k + ic_k$, show that

   $$f(\theta) = \frac{1}{2}B_0 + \sum_{n=1}^{\infty} B_n \cos(n\theta) + C_n \sin(n\theta),$$

   where $B_n = \frac{1}{2}(b_n + b_{-n})$ and $C_n = \frac{1}{2}(c_{-n} - c_n)$.

4. Convert your answer from (2) into an integral for B_n and C_n in terms of $f(\theta)$.

To complete our derivation, we still need to prove that a) any periodic real function f has a suitable F, and b) that F has a Laurent series which converges on the unit circle in $\mathbb{C}$. Unfortunately, you will have to wait until your complex analysis course!

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The Basel Problem

The *Basel problem*, first posed in 1644, is the challenge of summing up all the reciprocal squares:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

The first person to solve it was Leonhard Euler (1707–1783). Here, you will get a chance to explore his beautiful if non-rigorous method.

First, recall that we can write a finite polynomial $p(x)$ as a product of its roots $\{a_1, a_2, \ldots, a_n\}$:

$$p(x) = C(x - a_1)(x - a_2) \cdots (x - a_n) = C \prod_{i=1}^{n} (x - a_i).$$

The constant C is the coefficient of its leading term, Cx^n. Similarly, we can try to write $\sin(x)$ as a product of terms corresponding to its zeros at $\pi k \in \mathbb{Z}$:

$$\sin(x) = C \prod_{k\in\mathbb{Z}} (x - \pi k) = Cx \prod_{k\geq 1} (x - \pi k)(x + \pi k) = Cx \prod_{k\geq 1} (x^2 - \pi^2 k^2). \qquad (1)$$

We see immediately that there is a problem with this expression: the terms blow up as $k \to \infty$ for any fixed x! Let's try to diagnose it a little more carefully.

1. Take the log of both sides of (1), and apply the divergence test to the RHS. When does the series converge?

2. We can fix the problem in (1) by replacing the factors:

   $$x^2 - \pi^2 k^2 \to f(x)[x^2 - \pi^2 k^2],$$

   where $f(x)$ is a function with no zeros. Using the divergence test again, argue that $f(x) = 1/x^2$. For some new constant C', our heuristic factorisation becomes

   $$\sin(x) = C' x \prod_{k\geq 1} \left(1 - \frac{x^2}{\pi^2 k^2}\right). \qquad (2)$$

3. Divide both sides of (2) by x. Take the limit $x \to 1$, and use L'Hôpital's Rule (or standard limits) for the LHS, to argue that $C' = 1$.

4. Euler's ingenious trick was to Taylor expand both sides of (2),

   $$\sin(x) = x \prod_{k\geq 1} \left(1 - \frac{x^2}{\pi^2 k^2}\right) = \sum_{k\geq 0} c_k x^k, \qquad (3)$$

   and identify the coefficients of x^2.

(a) Explain why $c_2 = -1/6$. *Hint*: Taylor series.

(b) Prove using induction that

$$\prod_{k=1}^{n}(1 - a_k x) = 1 - x\sum_{k=1}^{n} a_k + O(x^2). \tag{4}$$

(c) Ignoring the question of convergence, take $n \to \infty$ in the previous question, apply to (3), and by identifying Taylor coefficients, deduce that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

5. **Bonus.** We can perform a similar trick for higher powers.

(a) Define the formal power series

$$\prod_{k\geq 1}(1 + a_k x) = \sum_{n\geq 0} A_n x^n. \tag{5}$$

Clearly, $A_0 = 1$, and you proved that

$$A_1 = \sum_{k\geq 1} a_k.$$

Prove that

$$A_n = \sum_{k_1<k_2<\cdots<k_n} a_{k_1} a_{k_2} \cdots a_{k_n}. \tag{6}$$

(b) Define

$$A_n^{(p)} = \sum_{k_1<k_2<\cdots<k_n} (a_{k_1} a_{k_2} \cdots a_{k_n})^p. \tag{7}$$

Show that

$$A_2 = \frac{1}{2}(A_1^2 - A_1^{(2)}).$$

(c) Identifying c_4 in the sine Taylor series and product expansion of (3), use the previous result to show that

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}.$$