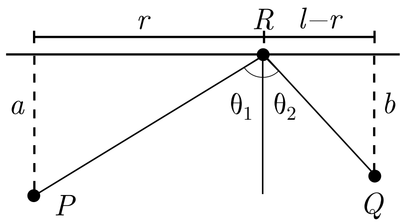
PHYC20014 Physical Systems

Classical Mechanics: Tutorial 1 Supplementary Problems

1. The Variational Biathlon. The Variational Biathlon is held at the beach and consists of two events. In Event 1, participants run as quickly as possible between two designated points on the sand, touching the shoreline en route. In Event 2, they start on the beach and race to a target point in the water using a combinaton of swimming and running. Athletes run at speed v_1 on the sand, and swim at speed v_2 in the water. You will determine the winning strategies!

(a) Argue that for trips confined entirely to one medium (sand or water), athletes should travel in straight lines.

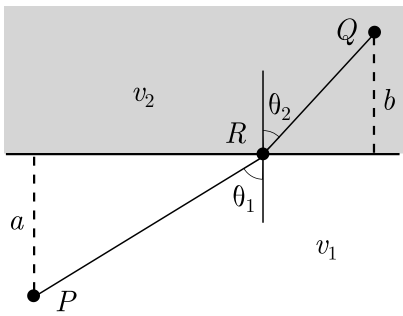


(b) In the first event (see above), athletes run from $P(0, a)$ to $Q(\ell, b)$, touching the straight shoreline ($y = 0$) in between. Let $R(r, 0)$ denote the point they touch the shoreline, with r to be determined. Show that the travel time is minimised when

$$\frac{r}{\sqrt{a^2 + r^2}} = \frac{\ell - r}{\sqrt{b^2 + (\ell - r)^2}}. \tag{1}$$

(c) Dusting off your high school trig, show that (1) implies $\theta_1 = \theta_2$ in the figure above.

(d) Solve (c) using part (a) alone.



(e) Now consider the second event (pictured above). Athletes must get from point $P(0, a)$ on the beach to $Q(\ell, -b)$ in the water, crossing from beach to ocean at $R(r, 0)$ for some r. Show that their travel time is minimised when

$$\frac{r}{v_1\sqrt{a^2 + r^2}} = \frac{\ell - r}{v_2\sqrt{b^2 + (\ell - r)^2}}, \tag{2}$$

and hence deduce that

$$v_1^{-1} \sin\theta_1 = v_2^{-1} \sin\theta_2.$$

2. POLA and mechanical energy.* The mechanical energy of the *true* path of a system can be calculated from the action in a slightly unexpected way. This insight turns out to be quite deep — in fact, it points the way to quantum mechanics — but we won't explore that here.

(a) Show from the POLA that the total mechanical energy E at the start of the trajectory satisfies

$$\frac{\partial S_{\text{true}}}{\partial t_1} = E(t_1)$$

for a general one-dimensional Lagrangian

$$L = \frac{1}{2}m\dot{x}^2 - V(x).$$

HINT. Consider separate infinitesimal changes to the starting time $t_1 \to t_1 + \delta t$ and path $x(t) \to x(t) + \delta x(t)$, where δx vanishes at t_2 but not necessarily at t_1. Combine your results using the relation between total and partial derivatives of S_{true}.

(b) Confirm the result in (a) explicitly for the action S_{true} in the tennis ball problem.

(c) What about S_{true} for a free particle? This is subtle, so be careful!

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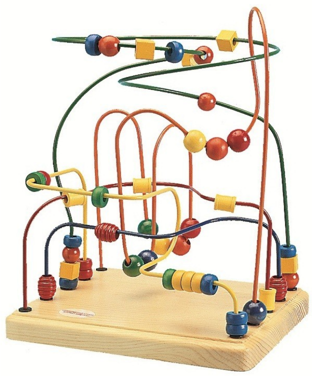
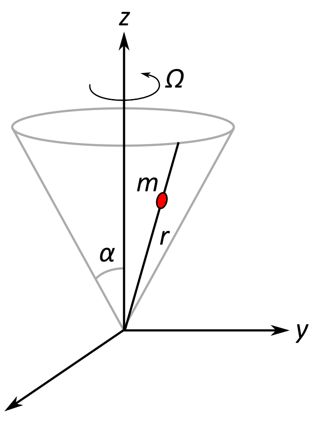
Classical Mechanics: Tutorial 2 Supplementary Problems

1. Bead on a Wire. In this problem you'll play around with a simple bead and wire system—much simpler than the systems (below left) you may have enjoyed as a child! In our simple system, a wire emanates from the origin, making an angle α with the z-axis and rotating around it with angular velocity Ω. A bead of mass m is is free to slide up and down the wire, subject only to gravity.





(a) Calculate the Lagrangian for the bead in terms of r, the distance from the bead to the origin. You should find that

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\Omega^2 \sin^2\alpha) - mgr\cos\alpha. \tag{1}$$

(b) Use Lagrange's equation to deduce the equation of motion

$$\ddot{r} = r\Omega^2 \sin^2\alpha - g\cos\alpha. \tag{2}$$

(c) Verify that (2) is solved by

$$r(t) = A\sinh(kt) + B\cosh(kt) + \frac{g\cos\alpha}{k^2}, \tag{3}$$

where $k = \Omega\sin\alpha$ and A and B are constants of integration related to the initial conditions $r(0), \dot{r}(0)$ by

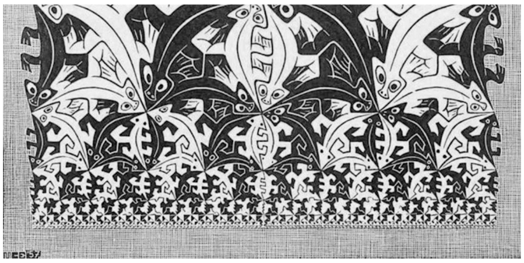
$$A = \frac{\dot{r}(0)}{k}, \quad B = r(0) - \frac{g\cos\alpha}{k^2}. \tag{4}$$

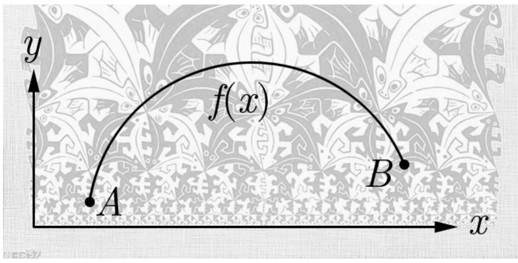
(d) At $t = 0$, the bead is launched from the origin up the wire at speed v. Assuming that $\alpha < \pi/2$, show that the bead never returns to the origin as long as

$$v \geq \left(\frac{g}{\Omega}\right)\cot\alpha.$$

HINT. You can either use (3) to evaluate $\dot{r}$ directly, or conservation arguments from the effective potential in (1).

2. Hyperbolic Lizards. One morning, you wake from troubled dreams to find yourself trapped in the Escher picture below, where distance is measured in lizards.[1] As you move towards the bottom of the picture, the lizards get smaller; equivalently, a rigid object (like a ruler) gets longer in lizard units.





2D lizard space has coordinates (x, y). At height y, the number of lizards in an interval with small coordinate displacements Δx and Δy is

$$\Delta \ell = \frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{y}.$$

The number of lizards encountered on a path $\mathcal{P}$ (parameterised by $y = f(x)$, $x \in [x_1, x_2]$) is therefore

$$\ell(\mathcal{P}) = \int_{x_1}^{x_2} \frac{\sqrt{1 + f'(x)^2}}{f(x)}\, dx \equiv \int_{x_1}^{x_2} L(f(x), f'(x), x)\, dx. \tag{5}$$

In the interests of encountering as few lizards as possible on a trip from $A(x_1, y_1)$ to $B(x_2, y_2)$, you can minimise the length of your path using Lagrange's equations.

(a) We have deliberately written (5) to remind you of the Principle of Least Action, with x playing the role of t and $f(x)$ the role of $x(t)$. Using Lagrange's equation, show that the length is minimised for a function f satisfying

$$ff'' = -[1 + (f')^2]. \tag{6}$$

(b) Check that

$$f(x) = \sqrt{R^2 - (x - k)^2} \tag{7}$$

satisfies (6). In other words, to minimise the lizards you step on, travel along arcs of circles centred on the boundary. The x-coordinate of the centre k and the radius R can always be chosen to match the endpoints A and B.[2]

[1] Apologies for Mark Van Raamsdonk for "borrowing" this joke.

[2] Well, sort of. For the case where $x_1 = x_2$, $y_1 \neq y_2$, the shortest path is a vertical line, which we can interpret as an arc on a circle with infinite radius and centre at infinity.

(c) Verify that, for a solution of (6),

$$c = L - f'\frac{\partial L}{\partial f'}$$

is a constant of motion, i.e. $c' = 0$. In terms of (7), what does c mean geometrically?

(d) *In 3D lizard space, we change to coordinates (x, y, z), with the bottom of the picture at $z = 0$. Then, for a path $\mathcal{P}(s) = (x(s), y(s), z(s))$ parameterised by a variable $s \in [s_1, s_2]$, the length in lizards is

$$\begin{aligned}\ell(\mathcal{P}) &= \int_{s_1}^{s_2} \frac{\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}}{z} \\ &= \int_{s_1}^{s_2} \frac{\sqrt{[x'(s)]^2 + [y'(s)]^2 + [z'(s)]^2}}{z(s)}\, ds \\ &\equiv \int_{s_1}^{s_2} L(x, y, z, x', y', z', s)\, ds.\end{aligned}$$

What are the ignorable coordinates and the corresponding conserved quantities? Without solving Lagrange's equations, briefly discuss what the conserved quantities imply about lizard-minimising paths, and relate this to the situation in 2D.

3. POLA and the Free Particle. In Tutorial 1, you proved that

$$\frac{\partial S_{\text{true}}}{\partial t_1} = E(t_1), \tag{8}$$

where S_{true} is the true action, t_1 is the *initial* time, and $E(t_1)$ is the total mechanical energy. But when we apply this to a free particle, with $S_{\text{true}} = mv^2(t_2 - t_1)/2$, we seem to get the wrong sign:

$$\frac{\partial S_{\text{true}}}{\partial t_1} \overset{?!}{=} -\frac{1}{2}mv^2 = -E(t_1).$$

Explain why there is actually no problem, and equation (8) is correct in this case too.

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Classical Mechanics: Tutorial 3 Supplementary Problems

1. Orbits around a black hole. Recall that for the Kepler problem (satellite of mass m orbiting a star of mass $M \gg m$), the equation of motion was

$$\frac{1}{2}\dot{r}^2 + V_{\text{eff,N}}(r) = E, \tag{1}$$

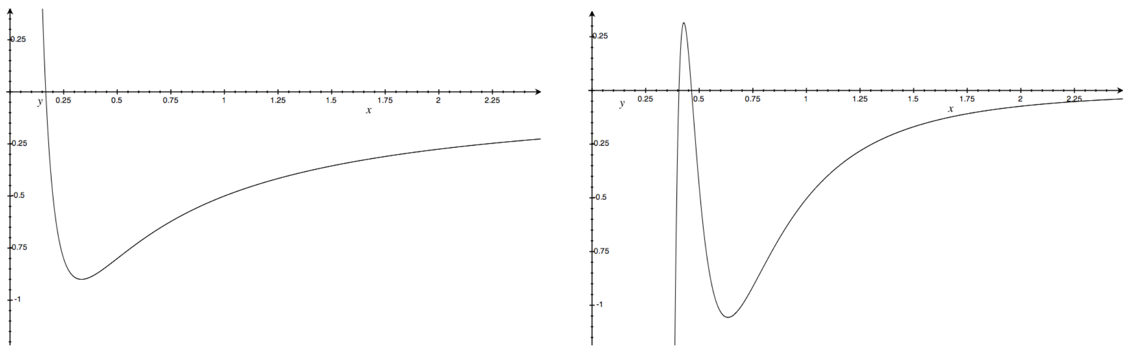
where E is the total energy of the planet and $V_{\text{eff,N}}$ the *Newtonian effective potential*:

$$V_{\text{eff,N}}(r) = \frac{J^2}{2r^2} - \frac{GM}{r}, \quad J = r^2\dot{\phi}.$$

Both J and E are constants of motion. For an orbit around a black hole, the equation of motion is the same, but the effective potential picks up a correction from general relativity:

$$V_{\text{eff}}(r) = \frac{J^2}{2r^2} - \frac{GM}{r} - \frac{GMJ^2}{c^2r^3}.$$

Here, c is the speed of light. Typical effective potentials are shown below, Newtonian on the left for comparison and black hole on the right:



(a) For the black hole, describe different possible motions of the satellite and how they depend on total energy E. Describe the qualitative difference between a Newtonian star and a black hole at small r.

(b) In terms of V_{eff}, what is the condition for a circular orbit at $r = r_0$? Show that r_0 must satisfy

$$GMr_0^2 - J^2 r_0 + \frac{3GMJ^2}{c^2} = 0. \tag{2}$$

Conclude that there are no circular orbits unless $J \geq \sqrt{12}GM/c$. What happens to the graph of V_{eff} as J is lowered past $\sqrt{12}GM/c$? From now on, we assume that $J > \sqrt{12}GM/c$ so the graph above is indeed representative.

(c) Consider a small perturbation to a circular orbit,

$$r(t) = r_0 + \xi(t). \tag{3}$$

We would like to determine the equation of motion for ξ (an equation for $\ddot{\xi}$) and thereby learn the fate of the perturbation.

To find the equation, differentiate (1) with respect to time, and substitute the perturbed orbit (3) in to find an equation for $\ddot{\xi}$:

$$\ddot{\xi}(t) + V'_{\text{eff}}(r_0 + \xi) = 0.$$

Now expand $V'_{\text{eff}}(r_0 + \xi)$ as a Taylor series in ξ, and since ξ is small, throw away terms which are second order or higher. You should obtain

$$\ddot{\xi}(t) + V''_{\text{eff}}(r_0)\xi(t) = 0. \tag{4}$$

(d) Explain how solutions to (4) depend on the sign of $V''_{\text{eff}}(r_0)$. Using (2), show that the orbit is stable (the perturbation does not grow exponentially) provided

$$r_0 > \frac{6GM}{c^2}.$$

(e) Calculate the period T of the circular orbit. You should find that

$$T = 2\pi r_0 \sqrt{\frac{r_0 - 3GM/c^2}{GM}}.$$

Show that in the limit $r_0 \gg 3GM/c^2$, this matches Kepler's third law.

HINT. Use (2) and $J = r_0^2\dot{\phi}$.

(f) For the stable perturbation in (d), what is the period of the oscillation of ξ compared to the period of the orbit? Draw the orbit. Is it closed?

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Classical Mechanics: Tutorial 4 Supplementary Problems

1. Mechanics of Mercury.[1] The orbital mechanics of the planet Mercury are truly fascinating. The planet is in a 3:2 spin-orbit resonance, meaning that it rotates exactly three times for every two revolutions that it makes around the Sun. Similarly, the Moon is in a 1:1 spin-orbit resonance; it rotates once for every revolution it makes around the Earth, which is why we always see the same face.

Mercury is not perfectly spherical. Hence, the Sun exerts a gravitational torque upon it, keeping it locked in its 3:2 spin-orbit resonance. (Convince yourself that this torque vanishes for a spherical body.) Modern experiments[2] have measured Mercury's moment-of-inertia tensor to about three significant figures. They show that Mercury is slightly triaxial, with principal moments satisfying $I_1 < I_2 < I_3$, and

$$\frac{I_2 - I_1}{I_3} = 2.2 \times 10^{-4} \ . \tag{1}$$

See Margot, J.-L. et al. 2012, *J. Geophys. Res.*, 117, E00L09, if you want to learn more!

In this question, we calculate the potential energy U of the gravitational interaction between Mercury and the Sun, which leads to the above torque. As we learned in class, U enters into the Lagrangian for Mercury's motion.

(a) Treat the Sun as a point mass $M_\odot$ at the origin. By dividing Mercury into infinitesimal pieces, show that the gravitational potential energy is given exactly by

$$U = -GM_\odot \int \frac{d^3\mathbf{x}'\, \rho(\mathbf{x}')}{|\mathbf{x}'|} \tag{2}$$

where the integral is over the volume of the planet, ρ is the mass density, and $\mathbf{x}'$ is the displacement of an infinitesimal mass element from the Sun.

(b) Let $\mathbf{x}$ denote the displacement of the centre of mass of Mercury from the origin, and let $\mathbf{s}$ be the displacement of an infintesimal mass element from the centre of mass. Then $\mathbf{x}' = \mathbf{x} + \mathbf{s}$. Mercury is small compared to its distance from the Sun, so $|\mathbf{s}| \ll r = |\mathbf{x}|$. By Taylor expanding or otherwise, show that

$$\frac{1}{|\mathbf{x}'|} = \frac{1}{r} - \frac{\mathbf{n} \cdot \mathbf{s}}{r^2} + \frac{3(\mathbf{n} \cdot \mathbf{s})^2 - |\mathbf{s}|^2}{2r^3} \tag{3}$$

with $\mathbf{n} \equiv \mathbf{x}/r$.

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[1] This question was written by Andrew Melatos.

[2] These experiments use two techniques: tracking Mercury's spin by bouncing radar echoes off the surface, and mapping its gravitational field from the trajectory of the *MESSENGER* spacecraft. *MESSENGER* (MErcury Surface, Space ENvironment, GEochemistry, and Ranging) orbited Mercury from 2011 until 2015.

(c) Substitute (3) into (2) to obtain

$$U = -\frac{GM_\odot M_{\text{Mercury}}}{r} - \frac{GM_\odot}{2r^3} \int d^3\mathbf{s}\, \rho_{\text{CM}}(\mathbf{s})\, [3(\mathbf{n}\cdot\mathbf{s})^2 - |\mathbf{s}|^2] \tag{4}$$

where $\rho_{\text{CM}}(\mathbf{s}) \equiv \rho(\mathbf{x}+\mathbf{s})$. You may assume that $\rho_{\text{CM}}(\mathbf{s}) = \rho_{\text{CM}}(-\mathbf{s})$. From now on, we use ρ to refer to the centre of mass distribution ρ_{CM}.

(d) From the general definition of the moment-of-inertia tensor I, prove that

$$\int d^3\mathbf{s}\, \rho(\mathbf{s})\, [3(\mathbf{n}\cdot\mathbf{s})^2 - |\mathbf{s}|^2] = \text{Tr}(I) - 3\mathbf{n}^{\text{T}} I \mathbf{n}\ , \tag{5}$$

where Tr denotes the trace, superscript $^{\text{T}}$ denotes the transpose, and we treat $\mathbf{n}$, $\mathbf{s}$ as column vectors. You can prove this component-wise by brute force or some other way; there are many ways to reach the answer.

(e) Mercury is almost spherical, so we have $I_1 \approx I_2 \approx I_3$. Let ψ be the angle between $\mathbf{n}$ and the principal axis $\mathbf{e}_1$. Using equations (4) and (5), deduce that

$$U = -\frac{GM_\odot M_{\text{Mercury}}}{r} - \frac{3GM_\odot(I_1 - I_2)}{2r^3} \sin^2\psi\ . \tag{6}$$

2. The phase-locked trombonist. A rhythmically-challenged trombonist tends to be out of time with the band. The band (phase θ_B) has tempo ω_B:

$$\frac{d\theta_B}{dt} = \omega_B.$$

The trombonist (phase θ) has a natural tempo $\omega \neq \omega_B$. Despite their natural inclination, the trombonist wants to stay in phase with the band, so θ evolves according to

$$\frac{d\theta}{dt} = \omega + I\sin(\theta_B - \theta), \tag{7}$$

where I measures the strength of the trombonist's response to the band. You can think of $\theta = 0,\ 2\pi,\ 4\pi,\ \ldots$ as the trombonist's beats, and $\theta_B = 0,\ 2\pi,\ 4\pi,\ \ldots$ as the band's beats. The phase difference $\psi \equiv \theta_B - \theta$ therefore satisfies

$$\frac{d\psi}{dt} = \omega_B - \omega - I\sin\psi. \tag{8}$$

(a) Explain why (7) pushes θ towards θ_B.

(b) Introducing variables $\tau = It$, $\delta = (\omega_B - \omega)/I$, show that (8) can be written

$$\frac{d\psi}{d\tau} = \delta - \sin\psi. \tag{9}$$

(c) Steady state solutions satisfy $d\psi/d\tau = 0$. This means the trombonist is *phase-locked* with the band—the tempos are the same, although their beats may be out of sync by a constant amount. Show that phase locking is only possible for

$$\omega - I \leq \omega_B \leq \omega + I.$$

When do the beats coincide?

(d) Even when phase-locked, the trombonist occasionally wanders out of phase a little. Whether they *stay* phase-locked depends on whether the solution is stable. Show that for $|\delta| < 1$ there are two solutions $\psi_1 < \psi_2$, with ψ_1 stable and ψ_2 unstable.

(e) For $|\delta| > 1$, the trombonist undergoes *phase drift*: the phase difference ψ inexorably changes with time. Show that the time it takes for ψ to change by 2π is

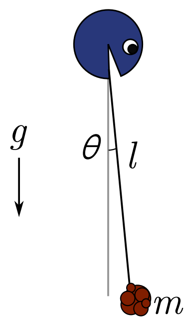
$$T = \frac{2\pi}{\sqrt{(\omega_B - \omega)^2 - I^2}}.$$

Put another way, T is the time it takes the band and trombonist to move a beat out of sync. This gives us a practical way to measure I, provided we know ω and ω_B.

HINT. You may need the integral

$$\int_0^{2\pi} \frac{d\psi}{\delta - \sin\psi} = \frac{2\pi}{\sqrt{\delta^2 - 1}}.$$

3. Spaghetti pendulum.* In an anarchic Carlton share house, spaghetti and meatballs are cooked in vast quantities each night and eaten communally. One of the housemates is a physics student, and as they slurp up a single spaghetto, they notice a meatball of mass m attached to the end.



To distract themselves from the meal, they begin speculating about the meatball's equation of motion.

(a) Write the Lagrangian for a free meatball subject to gravity. In terms of the (ℓ, θ) coordinates in the diagram, you should find

$$L = \frac{1}{2}m(\ell^2\dot{\theta}^2 + \dot{\ell}^2) + mg\ell\cos\theta.$$

Hence, derive the equations of motion

$$\ell\ddot{\theta} + 2\dot{\ell}\dot{\theta} + g\sin\theta = 0, \quad \ddot{\ell} = \ell\dot{\theta}^2 + g\cos\theta.$$

(b) Suppose the housemate slurps up a spaghetto at constant rate v. Show that for small oscillations of the meatball (i.e. small angles), the first equation of motion becomes

$$\ell\ddot{\theta} - 2v\dot{\theta} + g\theta = 0. \tag{10}$$

(c) Using the chain rule, rewrite (10) as

$$\ell\frac{d^2\theta}{d\ell^2} + 2\frac{d\theta}{d\ell} + \frac{g}{v^2}\theta = 0. \tag{11}$$

Now making the change of variables $x \equiv -2\sqrt{g\ell}/v$ and $y \equiv x\theta$, show that (11) becomes

$$x^2y'' + xy' + (x^2 - 1)y = 0 \tag{12}$$

where a dash denotes derivatives with respect to x. The differential equation (12) is solved by *Bessel functions* $J_1(x)$ and $Y_1(x)$:

$$y(x) = AJ_1(x) + BY_1(x).$$

(d) For simplicity, assume $A = 1$ and $B = 0$. Suppose the spaghetto has length L at time $t = 0$. Revert to our original variables, and express θ as a function of t. If you have a computer handy, graph θ and give a qualitative description of the dynamics of the meatball.

PHYC20014 Physical Systems

Classical Mechanics: Tutorial 5 Problems

1. Canonical transformations. In Lagrangian mechanics, we can choose coordinates at will; the Principle of Least Action (or equivalently, Lagrange's Equation) does not depend on how we label configuration space. Hamiltonian mechanics is a little bit different: Hamilton's equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{1}$$

are not invariant under any old change of phase space coordinates $(q_i, p_i) \to (Q_i(q,p), P_i(q,p))$. A coordinate transformation preserving (1) is called a *canonical transformation.*

(a) In lectures, we derived the Hamiltonian for a 2D simple harmonic oscillator. In 1D, the Hamiltonian is even simpler:

$$H(q,p) = \frac{p^2}{2m} + \frac{1}{2}kq^2 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2. \tag{2}$$

Rotate coordinates in phase space by an angle θ:

$$Q(q,p) = q\cos\theta - p\sin\theta, \quad P(q,p) = q\sin\theta + p\cos\theta.$$

Show that Q and P evolve according to

$$\dot{Q} = P\left[\frac{1}{m}\cos^2\theta + m\omega^2\sin^2\theta\right] + Q\cos\theta\sin\theta\left[m\omega^2 - \frac{1}{m}\right]$$
$$\dot{P} = -Q\left[\frac{1}{m}\sin^2\theta + m\omega^2\cos^2\theta\right] + P\cos\theta\sin\theta\left[\frac{1}{m} - m\omega^2\right].$$

Write down the Hamiltonian $H(Q,P)$ in the new coordinates. When is the transformation canonical?

(b) Suppose the transformation $(q_i, p_i) \to (Q_i, P_i)$ is canonical. Argue that the Poisson bracket is automatically preserved:

$$\{f,g\}_{(q,p)} \equiv \frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j} - \frac{\partial g}{\partial q_j}\frac{\partial f}{\partial p_j} = \frac{\partial f}{\partial Q_j}\frac{\partial g}{\partial P_j} - \frac{\partial g}{\partial Q_j}\frac{\partial f}{\partial P_j} \equiv \{f,g\}_{(Q,P)}.$$

Also derive the converse: a transformation preserving the Poisson bracket is canonical.

(c) Consider an infinitesimal change of coordinates,

$$q_i \to Q_i = q_i + \epsilon A_i(q,p)$$
$$p_i \to P_i = p_i + \epsilon B_i(q,p).$$

Show this transformation is canonical if there is some smooth function $G(q,p)$ satisfying

$$A_i = \frac{\partial G}{\partial p_i}, \quad B_i = -\frac{\partial G}{\partial q_i}.$$

This looks a lot like Hamilton's equations. In fact, setting $A_i = \dot{q}_i$, $B_i = \dot{p}_i$, and $G = H$, we see that evolving the system an infinitesimal time, $t \to t + \epsilon$, is canonical.

2. Action-angle variables. We now see how canonical transformations can make life simpler in Hamiltonian mechanics. For the 1D SHO, make the transformation $(q,p) \to (\theta, I)$ given by

$$q = \sqrt{\frac{2I}{m\omega}} \sin\theta, \quad p = \sqrt{2Im\omega}\cos\theta. \tag{3}$$

The new coordinates are called *action-angle variables*.

(a) Demonstrate that $(q,p) \to (\theta, I)$ is canonical using the Poisson bracket.

(b) Derive the Hamiltonian in the new variables. You should find

$$H(\theta, I) = \omega I.$$

(c) What are Hamilton's equations? Draw the corresponding trajectories in phase space.

(d) Consider a 1D Hamiltonian

$$H(q,p) = \frac{p^2}{2m} + V(q)$$

exhibiting periodic motion. The corresponding action variable I is the area of phase space enclosed in a single orbit divided by 2π:

$$I = \frac{A}{2\pi}.$$

Check that this agrees with (3).

3. Delays and social media. Consider a toy model of the popularity P of a topic on social media,

$$\dot{P}(t) = \dot{P}(t-T) + A[P(t) - P(t-T)]. \tag{4}$$

The first term on the right measures the response to trending, while the last two (governed by the "activity" A) correspond to random surfing onto new topics or away from old ones. Finally, $T > 0$ is the characteristic time lag for users to respond to new items.

(a) Trial an exponential $P(t) = e^{\lambda t}$ in (4). Show that λ must satisfy the equation

$$(\lambda - A)(1 - e^{-\lambda T}) = 0. \tag{5}$$

(b) Show that (5) has solutions $\lambda = A$ and $\lambda = im\omega$ for $\omega \equiv 2\pi/T$, $m \in \mathbb{Z}$.

(c) For $A > 0$, the solution $\lambda = A$ corresponds to runaway growth, i.e. something going *viral*. In terms of (4), explain qualitatively how this happens.

(d) The solutions $\lambda = im\omega$ correspond to periodic fluctuations in popularity. In terms of (4), explain qualitatively how oscillations arise. Evidently, for this model, trending alone cannot generate viral success.