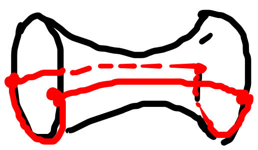
BLACK HOLE MICROSTATES VS. THE ADDITIVITY CONJECTURES

(HAYDEN & PENINGTON, 2012.0861)



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UBC STRINGS GROUP MEETING

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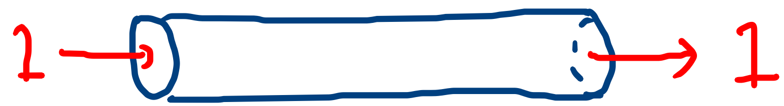
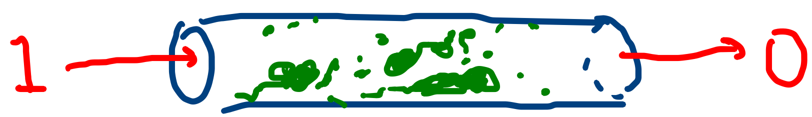
1. NOISY CLASSICAL CHANNELS

- A channel is pipe for moving bits around:



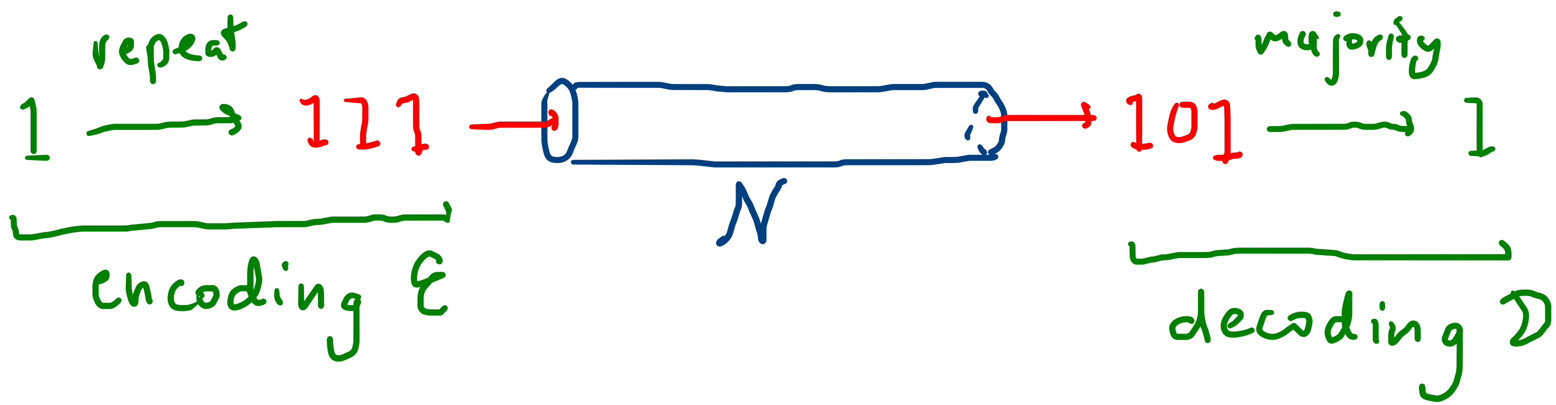
- It can be clean (noiseless) or dirty (noisy), with some probability of corrupting bits.



- We'll focus on dirty pipes, i.e. noisy channels.

- For a noisy channel, we may have to use it multiple times to reliably communicate one bit.

- Terminology: we map messages to codewords which can be reliably distinguished after passing through the pipe.

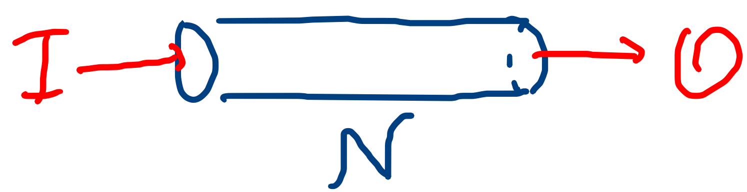


- A simple example is the repetition code. We encode the message (1) as a codeword (111). We decode using majority rule.

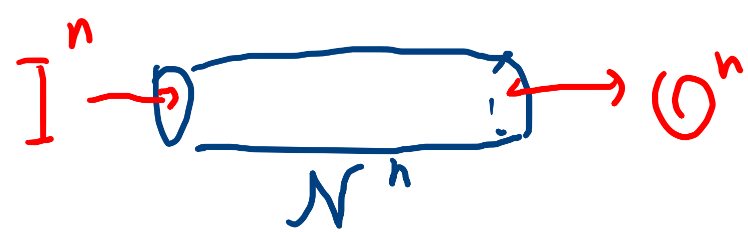
- The rate R of an encoding / decoding scheme is the number of bits per use.

$$1 \xrightarrow{\mathcal{E}} 111 \qquad R = \frac{1}{3}$$

- More formally, suppose our noisy channel $\mathcal{N}$ has a discrete input alphabet $\mathcal{I}$ and output alphabet $\mathcal{O}$:



For n channel uses,



- A rate R is achievable if there is a family of schemes $(\mathcal{E}_n, \mathcal{D}_n)$ which
  - send nR-bit messages
  - in n channel uses
  - with decoding error $o(n)$.

$$M \in 2^{nR} \xrightarrow{\mathcal{E}_n} I^n \xrightarrow{\mathcal{N}^n} O^n \xrightarrow{\mathcal{D}_n} M' \in 2^{nR}$$

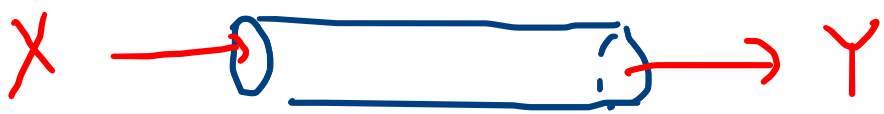
$$p(M \neq M') = o(n)$$

- The channel capacity $C(\mathcal{N})$ is the best achievable rate:

$$C(\mathcal{N}) = \sup_{(\mathcal{E}_n, \mathcal{D}_n)} R(\mathcal{E}_n, \mathcal{D}_n).$$

- This looks impossible to compute! But suppose our channel is memoryless, i.e. each use looks the same.

- If we think of the input as a random variable $X \in \mathcal{I}$, then the channel induces a random variable $Y \in \mathcal{O}$ at the other end:



$p(x)$ $\qquad$ $P(y|x)$ $\qquad$ $p(y) = \sum_x p(y|x)\,p(x)$

- In this case, we can avoid the horrible supremum!

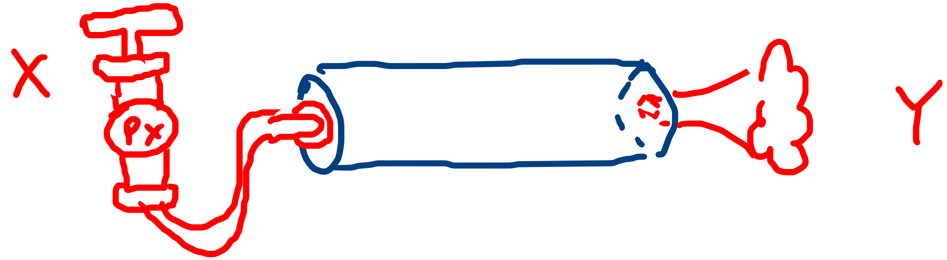
- Shannon's noisy channel coding theorem states that

$$C(\mathcal{N}) = \max_{p(x)} I(X:Y),$$

where $I(X:Y) = H(X) + H(Y) - H(XY)$.

- In a way, this is sensible, since $I(X:Y)$ tells us how many bits Y can extract from X.

- And because $\mathcal{N}$ is memoryless, we just pump as many bits through as possible per use.

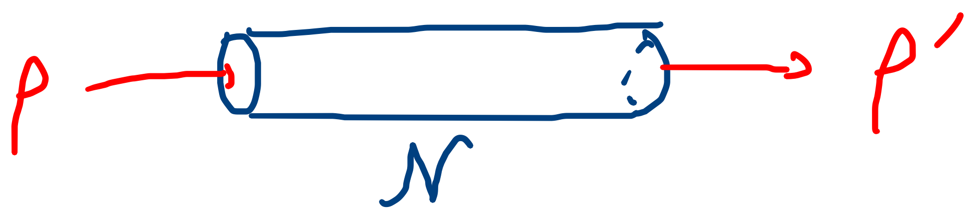


2. QUANTUM CHANNELS & ADDITIVITY

- In quantum mechanics, X becomes a density:

$$\rho_X = \sum_{x \in I} p(x) |x\rangle\langle x|, \qquad \langle x | x' \rangle = \delta_{xx'}.$$

- Instead of classical channels, I can ask about the classical capacity of a quantum channel:



for a memoryless CPTP operation $\mathcal{N}$.

- If we just input densities as products, i.e.

$$\rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_n$$

the Holevo - Schumacher - Westmoreland (HSW, 96/97) Theorem gives an analogue of Shannon:

$$C_{prod}(N) = \chi(N)$$

$$= \max_{\{\rho_i, p_i\}} \left[S\left(N\left(\sum_i p_i \rho_i\right)\right) - \sum_i p_i S(N(\rho_i)) \right]$$

where χ is Holevo information and S is EE.

- Like Shannon's theorem, this product state capacity is a tractable optimization problem.

- Importantly, in order to achieve the product state capacity, we must be allowed to use entanglement in our decoding scheme:

$$M \xrightarrow[\mathcal{E}_2]{} \rho_1 \otimes \rho_2 \xrightarrow[\mathcal{N}^{\otimes 2}]{} \rho_1' \otimes \rho_2' \xrightarrow[\mathcal{D}_2]{\text{ent.}} M'$$

- If entanglement in decoding is so helpful, what about entanglement in encoding?

$$M \xrightarrow[\mathcal{E}_2]{\text{ent.}} \rho_{12} \xrightarrow[\mathcal{N}^{\otimes 2}]{} \rho_{12}' \xrightarrow[\mathcal{D}_2]{\text{ent.}} M'$$

Is the unrestricted capacity $C(\mathcal{N})$ different?

- People played around with entangled inputs and found no improvement to achievable rates.
- This led Holevo (2007) to conjecture that

$$C_{prod}(\mathcal{N}) = C(\mathcal{N})$$

OR

$$\chi(\mathcal{N}_1 \otimes \mathcal{N}_2) = \chi(\mathcal{N}_1) + \chi(\mathcal{N}_2).$$

- Note: by treating the $n \to \infty$ limit as a single big space, HSW implies

$$C(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n} \chi(\mathcal{N}^{\otimes n})$$

$$\overset{\text{additivity}}{\Longrightarrow} \quad C(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n} \cdot n\chi(\mathcal{N}) = \chi(\mathcal{N}) = C_{prod}(\mathcal{N}).$$

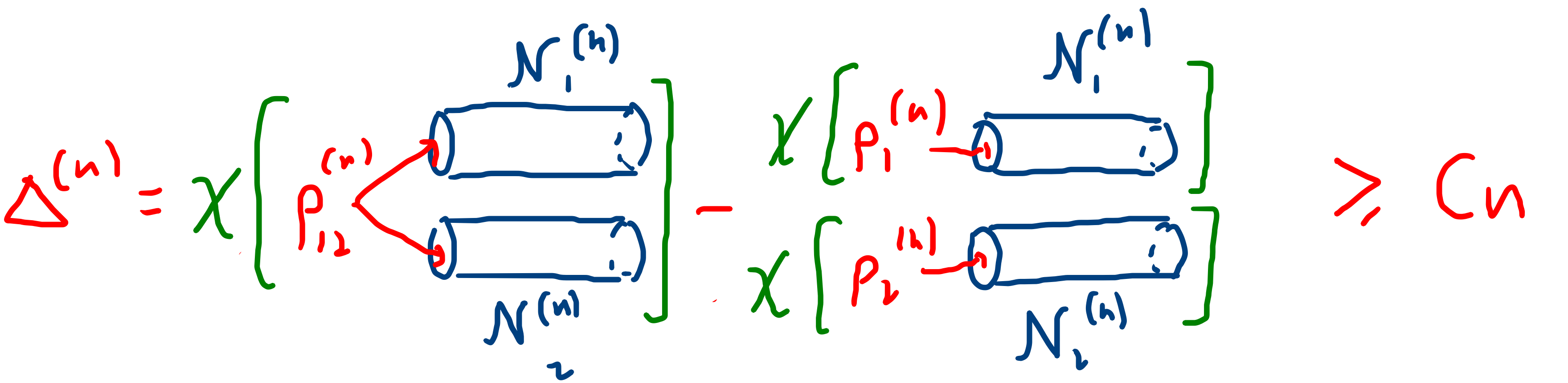
- It turns out that additivity conjecture is false!
  Hastings found examples where

$$\Delta = \chi(\mathcal{N}_1 \otimes \mathcal{N}_2) - \chi(\mathcal{N}_1) - \chi(\mathcal{N}_2) > 0 .$$

But they're tiny! Biggest $\Delta \leq \log 2$, i.e. one bit.

- Refinement: extensive additivity conjecture, no family of channels s.t.

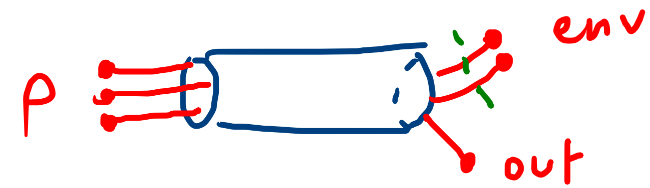
$$\mathcal{N}_{1,2}^{(n)} : A^{(n)} \to B^{(n)}$$



- Additivity of Holevo information turns out to be equivalent to many other additivity conditions.

- The minimum output entropy $S_{min}(\mathcal{N})$ is the smallest EE that pops out of the channel:

$$S_{min}(\mathcal{N}) = \inf_{\rho} S(\mathcal{N}(\rho)).$$



Intuitively, this is the minimum noise or environmental entanglement added by the channel.

- King & Ruskai (1999) showed additivity of χ iff

$$S_{min}(\mathcal{N}_1 \otimes \mathcal{N}_2) = S_{min}(\mathcal{N}_1) + S_{min}(\mathcal{N}_2).$$

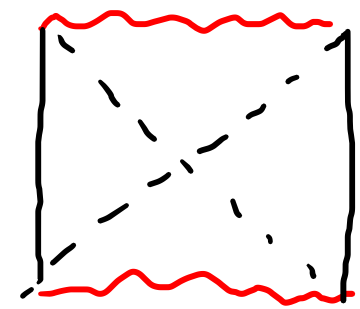
The extensive additivity form is also equivalent.

3. BLACK HOLES AND ADDITIVITY

- Let's head to the terra firma of black holes.
- We will consider a holographic CFT on a sphere S^d, with $G_N \to 0$.
- The thermofield double (TFD) on two copies of the sphere, with energy eigenstates $|n\rangle$, is

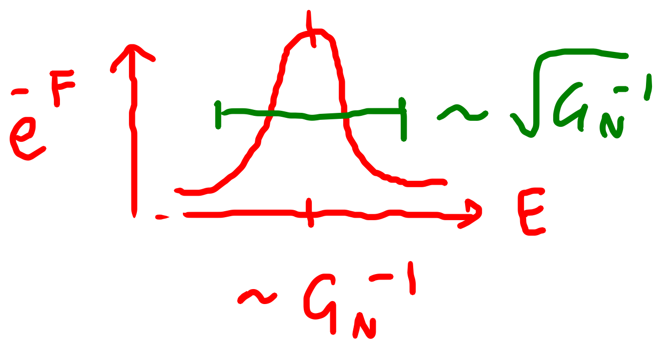
$$|TFD\rangle \propto \sum_n e^{-\beta E_n/2} |n\rangle |\bar{n}\rangle$$

CPT



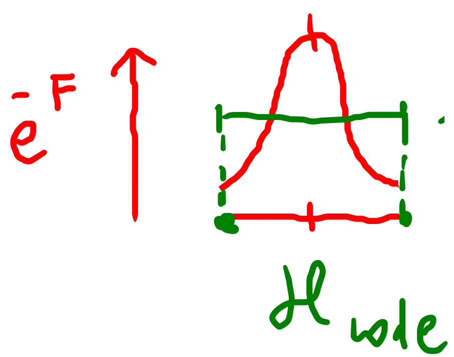
It is dual to the eternal AdS wormhole.

- Additivity is a finite-dimensional affair. To massage the TFD into this form, note that



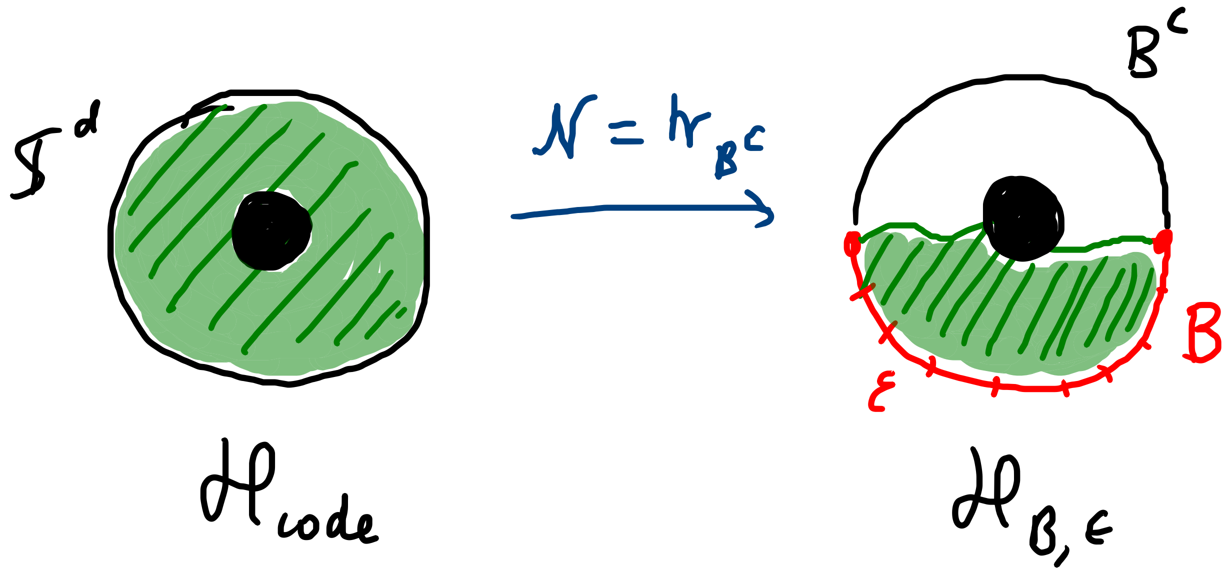
- saddle point energy $E \sim \frac{1}{G_N}$;
- energy spread $\Delta E \sim \sqrt{\frac{1}{G_N}}$.

- We can approximate the TFD on an energy band of width $O(\Delta E)$ centred at E, which will act as a "code subspace" $\mathcal{H}_{code}$:



$$|TFD'\rangle \propto \sum_{n \in \mathcal{H}_{code}} e^{-\beta E_n/2} |n\rangle|\bar{n}\rangle .$$

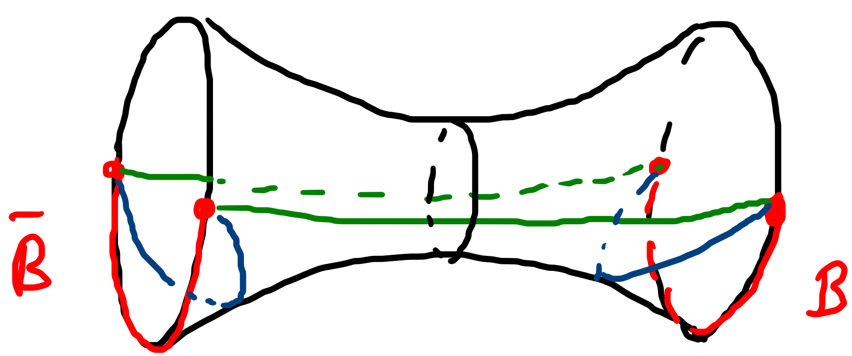
- Consider a quantum channel $\mathcal{N}$ which restricts states in $\mathcal{H}_{code}$ to a subregion $B \subseteq S^d$:



- To ensure we map to a finite-dimensional Hilbert space, we lattice regularize B.
- We expect this to be fine for spacing $\epsilon \ll E^{-1}$.

- In this setup, we will try to violate the extensive additivity conjecture for S_{min}.

- The idea is simple: in the TFD, make B large enough that the minimal surface homologous to $B \cup \bar{B}$ threads the wormhole:

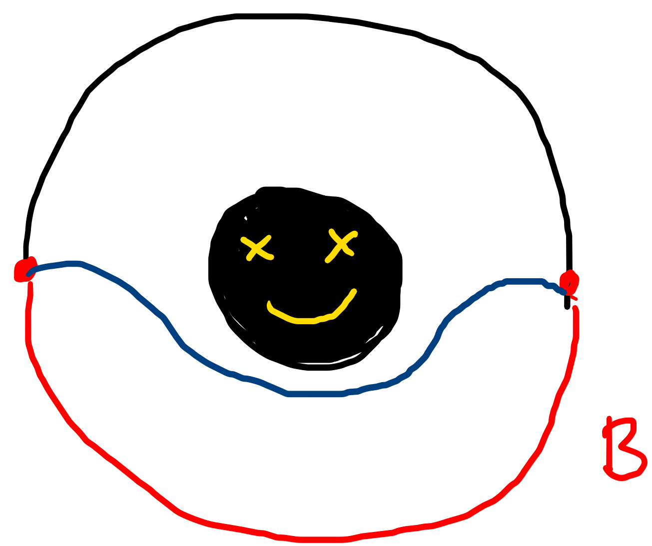


$2A_{dis} < 2A_B$

- By construction, this is smaller than the union of minimal surfaces homologous to $B, \bar{B}$.

- For a generic state ρ in $\mathcal{H}_{code}$, the EE of $\mathcal{N}(\rho)$ is given by the RT formula:

$$S(\mathcal{N}(\rho)) = S(\rho_B) = \frac{A_B}{4G_N} + O(1).$$



- We expect generic states in the energy band to have similar exterior geometries.

- Suppose every ρ is typical. Then

$$S_{min}(\mathcal{N}) = \inf_{\rho} S(\rho_B) = \frac{A_B}{4G_N} + O(1).$$

- To compute S_{min} in the doubled channel, use Fannes' inequality:

$$|S(\rho) - S(\sigma)| \leq \|\rho - q\|_1 \log \dim \mathcal{H} + o(\|\rho - q\|_1).$$

where $\|\cdot\|_1$ is trace distance.

- Since $1/G_N$ counts local DOFs, we have

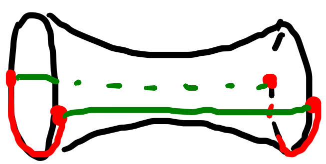
$$\log \dim \mathcal{H}_{B,\epsilon} \sim \frac{1}{G_N}.$$

- So, as long as $|TFD'\rangle$ approaches $|TFD\rangle$ in trace distance as $G_N \to 0$, the EE after the channel agrees at leading order:

$$G_N \left| S(\rho_{B\bar{B}}) - S(\rho'_{B\bar{B}}) \right| \lesssim \|\rho - \rho'\|_1 = o(G_N),$$

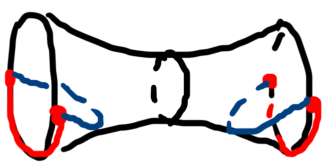
where $\rho = |TFD\rangle\langle TFD|$, $\rho' = |TFD'\rangle\langle TFD'|$.

- Now we get our violation of additivity:



$$S_{min}(N \otimes \bar{N}) \leq \frac{2A_{dis}}{4G_N} + O(1)$$

$$\vee \qquad\qquad \vee$$

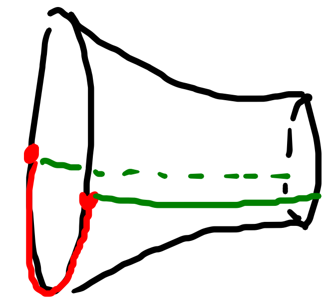


$$2 S_{min}(N) = \frac{2A_B}{4G_N} + O(1).$$

4. DISENTANGLED MICROSTATES

- Of course, we can easily avoid this conclusion if some atypical states $|\psi\rangle$ have

$$S(N(|\psi\rangle\langle\psi|)) = \frac{A_{dis}}{4G_N} + O(1).$$



(We can consider a pure state since these will minimize S_{min} by concavity of EE.)

- We call these microstates disentangled. Thus,

extensive additivity $\Rightarrow$ disentangled microstates.

- But we can go further and estimate how many such microstates exist.

- Let's go through $\mathcal{H}_{code}$ and throw away (i.e. project onto the orthogonal complement of) disentangled states satisfying

$$S(\rho_B) = \frac{A_{dis}}{4G_N} + O(1).$$

- Let $\mathcal{H}_{code}^{typ}$ be a "typicalized" subspace such that the restricted channel $\mathcal{N}^{typ}$ satisfies

$$S_{min}(\mathcal{N}^{typ}) - S_{min}(\mathcal{N}) \sim \frac{1}{G_N},$$

i.e. there are leading order shifts in S_{min}.

- If $|TFD'_{typ}\rangle$ still approached $|TFD\rangle$ in trace distance as $G_N \to 0$, we could still argue

$$S_{min}(N \otimes \bar{N}) = \frac{2A_{dis}}{4G_N} + O(1),$$

and we would violate extensive additivity.

- We're assuming we don't violate, so $|TFD'_{typ}\rangle$ is not a good approximation to $|TFD\rangle$:

$$\|\rho - \rho'_{typ}\|_1 \neq o(G_N).$$

This means there exists some projective measurement that distinguishes them with probability $O(1)$ even as $G_N \to 0$.

- Our measurement projects onto $\mathcal{H}_{dis}$, where

$$\mathcal{H}_{code} = \mathcal{H}_{dis} \oplus \mathcal{H}_{code}^{typ}.$$

- On $|TFD'\rangle$, it has $O(1)$ probability of success. Since projecting onto an individual state occurs with probability

$$p \sim e^{-S_{BH}}, \qquad S_{BH} \sim \log \dim \mathcal{H}_{code}.$$

- The dimension $N_{dis} = \dim \mathcal{H}_{dis}$ is then

$$O(1) \sim N_{dis}\, e^{-S_{BH}}$$

$$\Rightarrow \quad N_{dis} = O(e^{S_{BH}}) = e^{S_{BH} - O(1)}$$

This is an "almost" basis of disentangled states.

- One technical issue: we've ignored the energy spread, so $p \sim e^{-S_{BH}}$ isn't right.
- But even if the $\mathcal{H}_{dis}$ doesn't form an almost basis for the BH at inverse temp. β, it will be an almost basis for another BH!
- Let

$$p^{(\beta)}_{dis} = \langle \mathbb{P}_{dis} \rangle_\beta := \frac{\mathrm{Tr}(\mathbb{P}_{dis}\, e^{-\beta H})}{\mathrm{Tr}(e^{-\beta H})} .$$

This is the probability of projecting onto $\mathcal{H}_{dis}$.

- By assumption, $p^{(\beta)}_{dis} = O(1)$ for our BH.

- Now we vary wrt β to maximize $p_{dis}^{(\beta)}$:

$$\langle \mathbb{P}_{dis} H \rangle_{\beta^*} = \langle \mathbb{P}_{dis} \rangle_{\beta^*} \langle H \rangle_{\beta^*} . \qquad (*)$$

Define a provavility distrivution

$$q_n \propto C^{-1} \langle n | \mathbb{P}_{dis} | n \rangle e^{-\beta^* E_n} , \qquad C = \mathrm{Tr}[\mathbb{P}_{dis} e^{-\beta^* H}] .$$

- We can calculate the dimension of $\mathcal{H}_{dis}$ as

$$\dim \mathcal{H}_{dis} = \sum_n \langle n | \mathbb{P}_{dis} | n \rangle$$

$$= C \sum_n q_n e^{\beta^* E_n}$$

Jensen's inequality

$$\geq C \exp[\beta^* \sum_n q_n E_n] .$$

- But
$$\sum_n q_n E_n = \frac{\sum_n \langle n | \mathbb{P}_{dis} H e^{-\beta^* H} | n \rangle}{\mathrm{Tr}[\mathbb{P}_{dis} e^{-\beta^* H}]} = \frac{\langle \mathbb{P}_{dis} H \rangle_{\beta^*}}{\langle \mathbb{P}_{dis} \rangle_{\beta^*}},$$
which by (*) is simply $\langle H \rangle_{\beta^*}$. Hence
$$\dim \mathcal{H}_{dis} \geq C e^{\beta^* \langle H \rangle_{\beta^*}}$$
Gibbs
$$= \mathrm{Tr}[\mathbb{P}_{dis} e^{-\beta^* H}] e^{S_{BH}(\beta^*) - \log Z(\beta^*)}$$
$$= \langle \mathbb{P}_{dis} e^{-\beta^* H} \rangle_{\beta^*} e^{S_{BH}(\beta^*)} = p_{dis}^{(\beta^*)} e^{S_{BH}(\beta^*)}.$$
- Since $p_{dis}^{(\beta^*)} = O(1)$ (we maximized it!) we see that $\mathcal{H}_{dis}$ is an almost basis for the BH at β^*.
- Note: β^* is $O(1)$ near β since it lives in $\mathcal{H}_{code}$.

5. GEOMETRY & HORIZONS

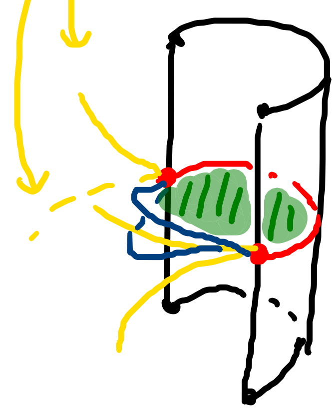
- There's no obvious reason for disentangled microstates to be geometric. But what if?
- Assume also they satisfy QES prescription:

$$S(\rho_B) = \min_{\gamma} \operatorname{ext} \Bigg[\overbrace{\frac{A(\gamma)}{4G_N} + S_{bulk}(R_\gamma)}^{S_{gen}} \Bigg]$$

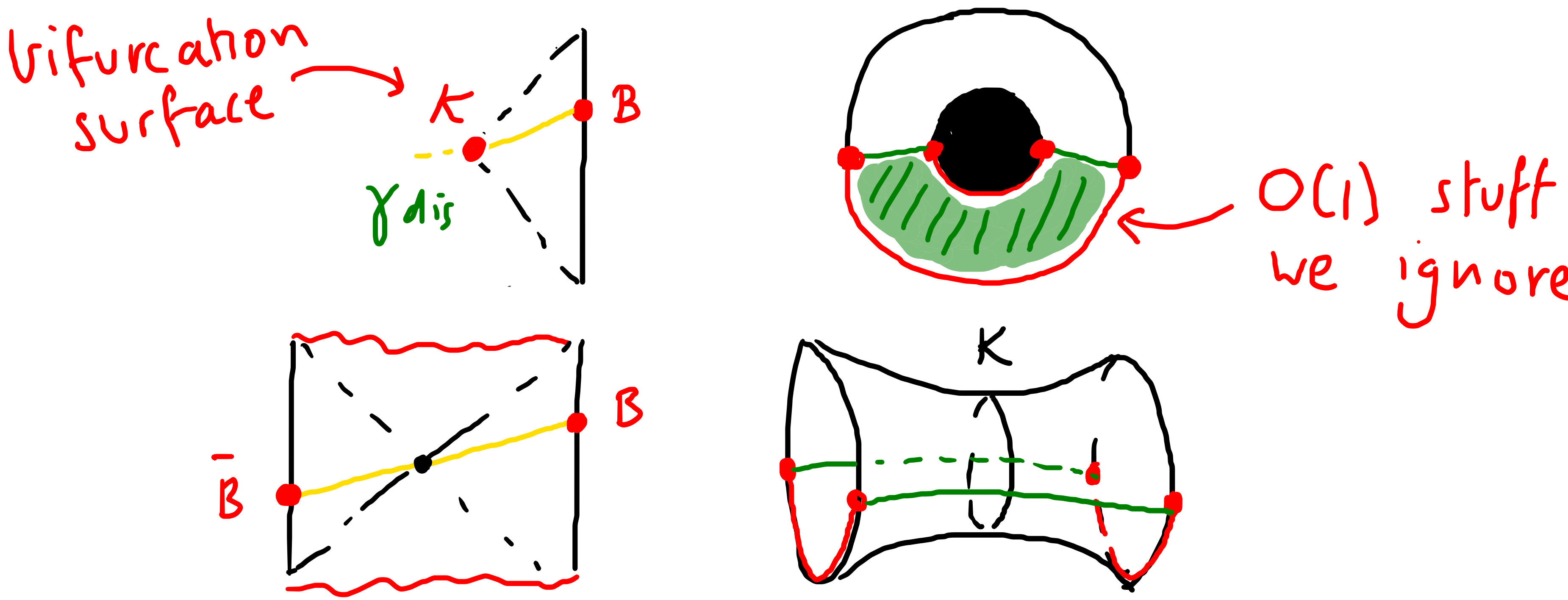
or maximin!

$$= \max_{C} \min_{\gamma \in C} \left[\frac{A(\gamma)}{4G_N} + S_{bulk}(R_\gamma) \right]$$

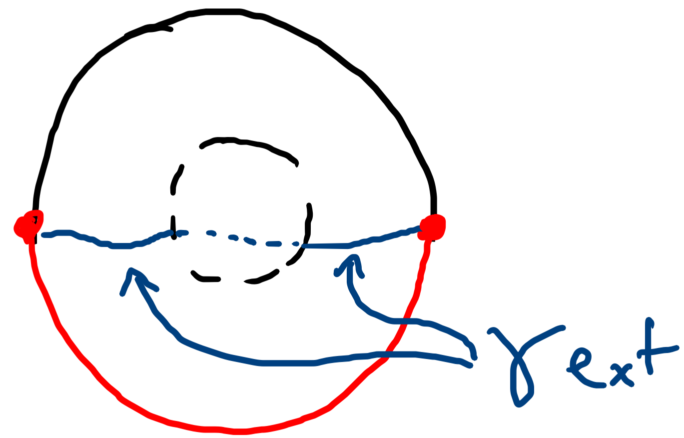
AdS-Cauchy slices



- Since we maximize over C, a lower bound on S_{gen} for some C applies to all C.
- Assume our microstate has an SAdS exterior.
- Adding some of the bifurcation surface κ, the minimal extremal surface is γ_{dis}:



- Now, for general γ homologous to B, let γ_{ext} be the part of γ in the exterior:



- If there is any smooth interior spacetime,

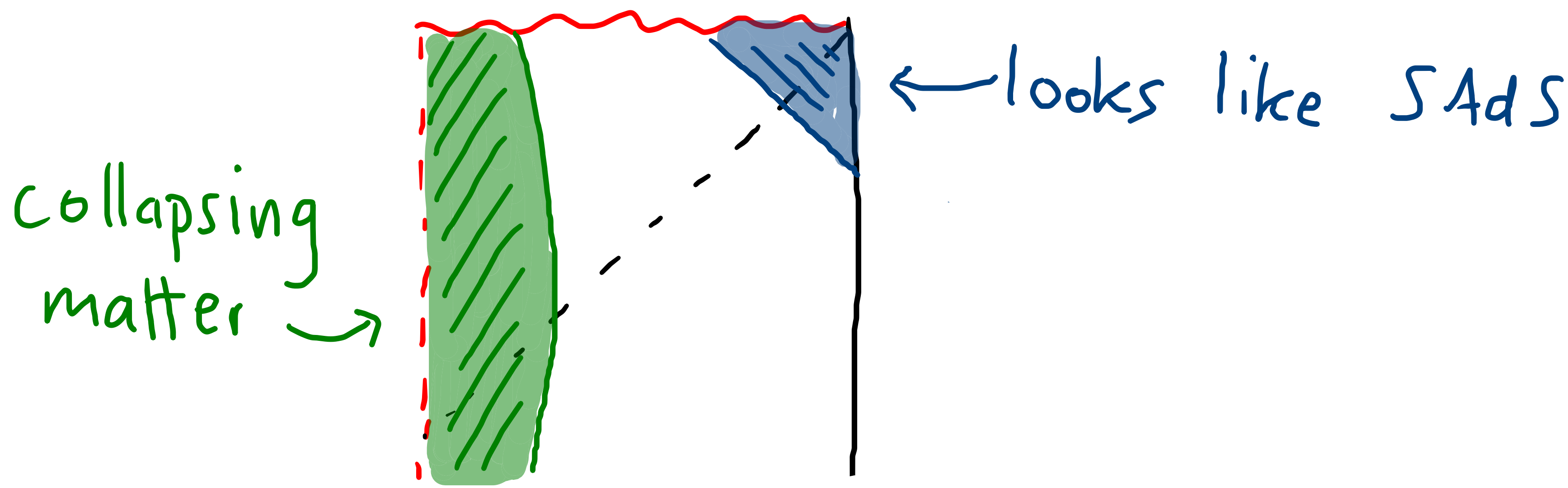
$$A(\gamma) > A(\gamma_{ext}) \geq A_{dis}.$$

$$\Rightarrow \qquad S_{gen}(\gamma) > \frac{A_{dis}}{4G_N} + O(1) = S(\rho_B).$$

- But $S_{gen}(\gamma)$ lower bounds the entropy! There cannot be a smooth interior.

- Of course, if a BH forms from collapse, it won't have a SAdS exterior.

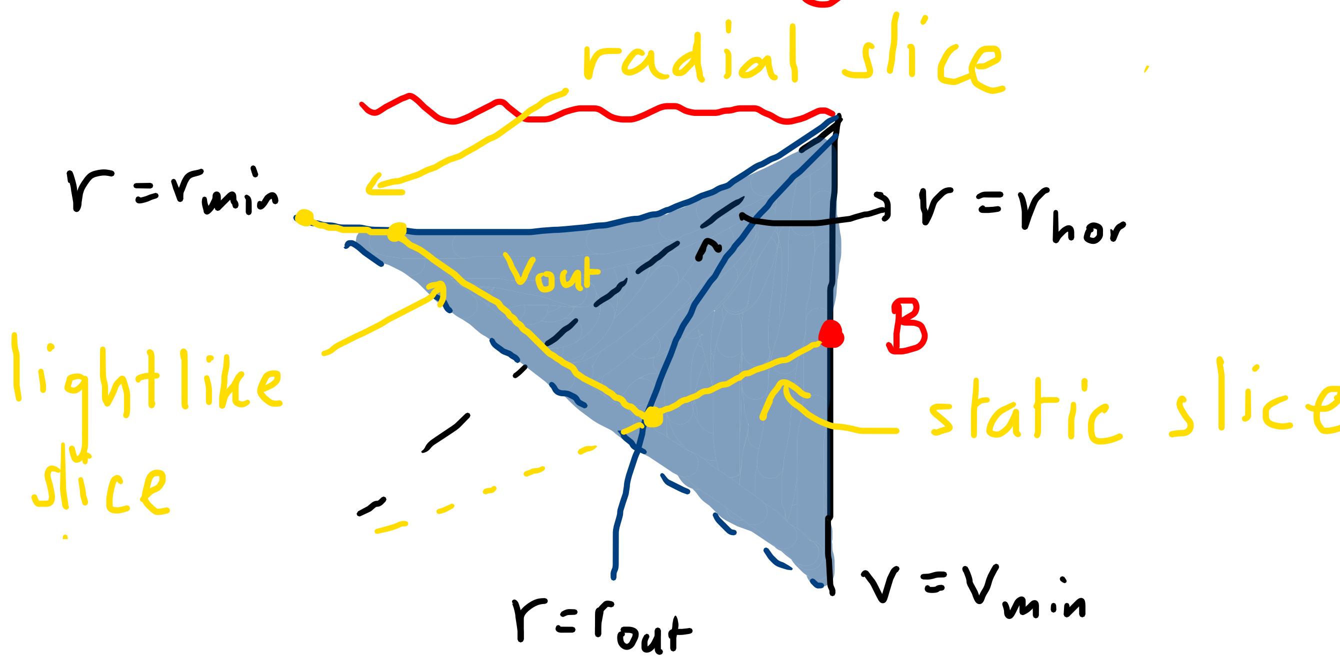
- But at late infalling times, it looks like SAdS:



- We assume we measure $S(\rho_B)$ well within the patch and the approximation works for

$$r > r_{min}, \quad v > v_{min}.$$

- We can draw our Cauchy slice as follows:



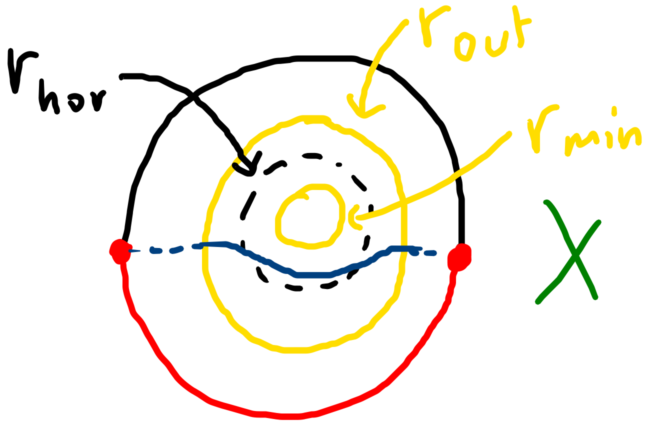
- The restriction γ_{ext} of any homologous γ to the static slice has minimum area

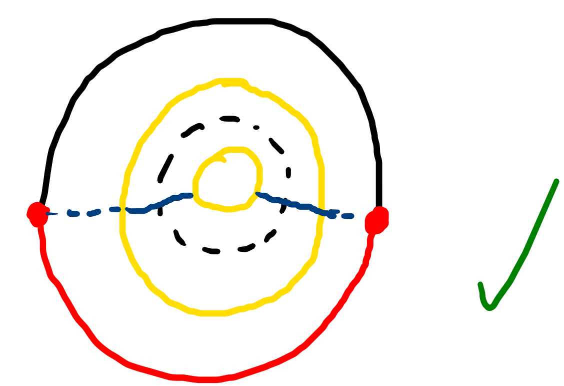
$$A(\gamma_{ext}) = A_{dis} - O\left(r_{hor}^{d-2}\sqrt{\beta(r_{out} - r_{hor})}\right),$$

which follows from linearizing the metric near r_{hor}.

- The correction is parametrically smaller than the horizon area $O(r_{hor}^{d-1})$ since $\beta \lesssim r_{hor}$.

- To avoid a paradox, namely $A(\gamma) > A_{dis}$, the the lightlike piece γ_{light} must be bounded parametrically by the correction.



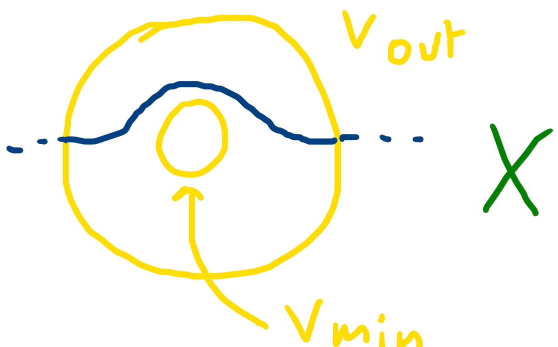


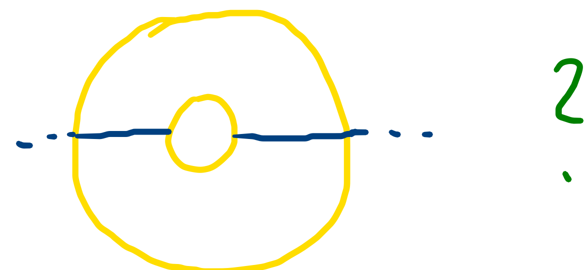
$A(\gamma_{light}) = O(r_{hor}^{d-1})$

$A(\gamma_{light}) = O(r_{hor}^{d-2}(r_{out} - r_{min}))$

- Since $r \simeq r_{hor}$ on γ_{light}, it must fall in to the sphere at $r = r_{min}$ to ensure this bound.

- This leaves an interior piece γ_{int}. It can either cover half the sphere at (r_{min}, v_{out}) or fall in again and connect to (r_{min}, v_{min}):

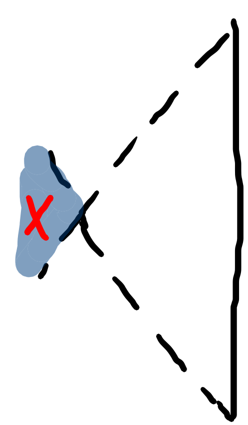




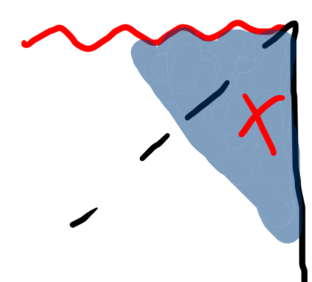
$$A(\gamma_{int}) = O\left(r_{hor}^{d-1}\right) \qquad A(\gamma_{int}) = O\left(r_{hor}^{d-1}\,\Delta v\sqrt{\frac{r_{hor}-r_{min}}{\beta}}\right)$$

- Provided $\Delta v = v_{out} - v_{min} \gg \beta$, and $r_{hor} - r_{min}$ is bigger than $r_{out} - r_{hor}$, the second term is parametrically bigger than the γ_{ext} correction.
- If so, $A(\gamma)$ must be bigger than A_{dis}. Contradiction!

- So, this means that if a disentangled microstate is geometrical and obeys QES, then
  - if it has a SAdS exterior, there is no smooth geometry beyond k;
  - it cannot have a SAdS "corner" at late infalling times (+ constraints).



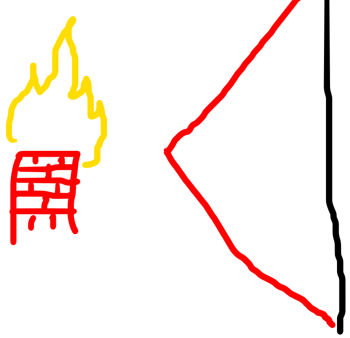
one exterior

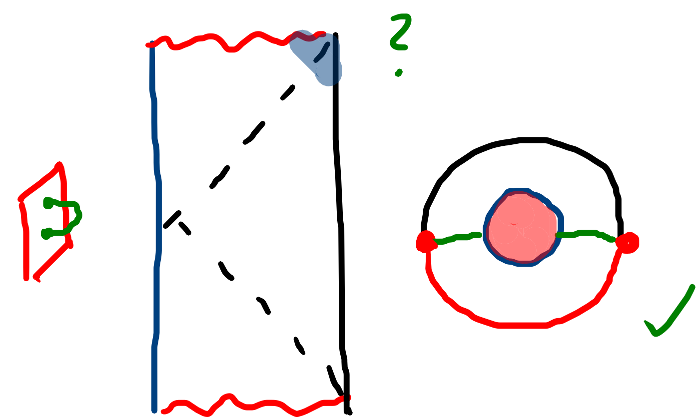


no corners

6. COMMENTS ON STRUCTURE

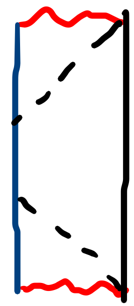
- The simplest possibility consistent with both rules is a firewall:



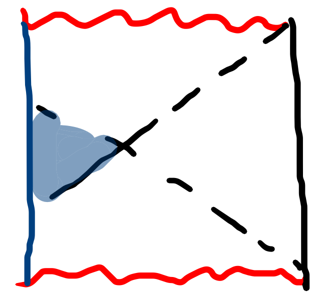


- They suggest that a better candidate is the $\mathbb{Z}_2$ quotient of SAdS, aka the $T=0$ brane. Since γ can end on the brane, QES gives A_{dis}.
- But why is the corner allowed?

- Let's ignore that, and ask: are there enough brane microstates to make an almost basis?



$T<0$ ✓



$T>0$ ✗

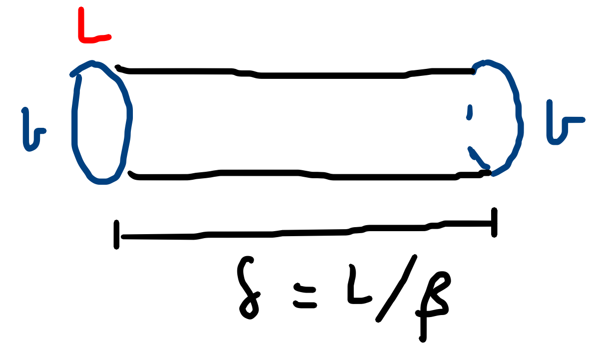
- Note that negative tensions are candidate disentangled states, but not positive tension.
- In a 2D CFT, $T<0$ microstate are given by

$$|\Psi_b\rangle \propto e^{-\frac{\beta}{4}H_c}|b\rangle, \quad g_b = \ln\langle 0|b\rangle < 0$$

for a boundary state $|b\rangle$ with entropy g_b.

- The associated entropy at small β is

$$S_b = \frac{\pi c L}{3\beta} + 2 g_b .$$



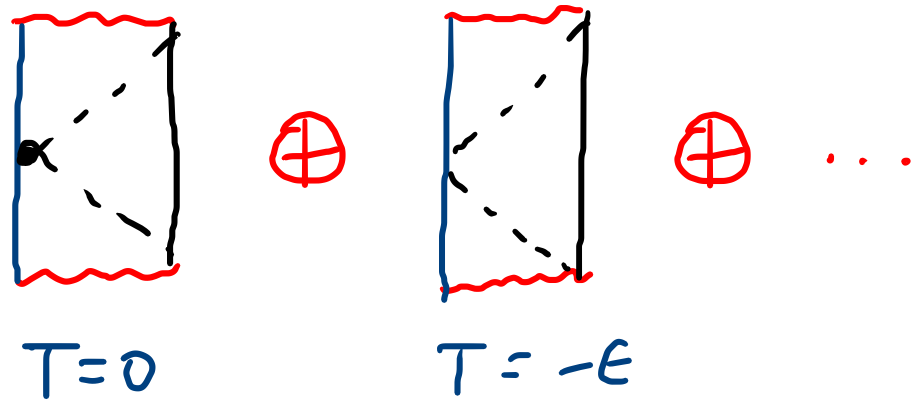
- The thermal entropy at β is

$$S_{th} = \frac{2\pi c L}{3\beta} .$$

For negative tensions, $S_b \leq \frac{1}{2} S_{th}$.

- Cf. Takayanagi et al., 2103.06893.

- So no single tension will suffice. But we're allowed different tensions!

$$\mathcal{H}_{dis} \overset{?}{=}$$ 

We would need a high "tension density" $O(e^{S_{BH}/2})$ to get an almost basis.

- These are preliminary ideas. Feedback welcome!

THANKS FOR LISTENING!